

Entanglement distillation by dissipation and continuous quantum repeaters

Karl Gerd H. Vollbrecht¹, Christine A. Muschik¹, and J. Ignacio Cirac¹

¹*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Strasse, D-85748 Garching, Germany*

Even though entanglement is very vulnerable to interactions with the environment, it can be created by purely dissipative processes. Yet, the attainable degree of entanglement is profoundly limited in the presence of noise sources. We show that distillation can also be realized dissipatively, such that a highly entanglement steady state is obtained. The schemes put forward here display counterintuitive phenomena, such as improved performance if noise is added to the system. We also show how dissipative distillation can be employed in a continuous quantum repeater architecture, in which the resources scale polynomially with the distance.

PACS numbers: 03.67.Ac, 03.67.Hk, 03.65.Ud

Entanglement plays a central role in applications of quantum information science such as quantum computation, simulation, metrology, and communication. However, any quantum technology is challenged by dissipation. The interaction of the system with its environment is regarded as a major obstacle, and in particular the degradation of entangled states due to dissipation is typically considered to be a key problem. Contrary to this belief, new approaches aim at utilizing dissipation for quantum information processes [1] including quantum state engineering [2–4], quantum computing [4], quantum memories [5], the creation of entangled states [6], and error correction [7].

Entanglement generated by dissipation has been demonstrated experimentally [8] following a recent theoretical proposal [6]. The main advantage of this scheme lies in the fact that entangled states are generated in a steady state. Furthermore, as opposed to standard methods, the desired state is reached independently of the initial one. By coupling two quantum systems to a common environment (e.g. the electromagnetic field [8]) a robust entangled steady state can be quickly generated and maintained for an arbitrary long time without the need for error correction such that entanglement is available any time.

As any other scheme, dissipative protocols are exposed to noise sources, which degrade the quality of the produced state and render it inapplicable for many important applications in quantum information, like quantum communication where noise effects increase dramatically

with the distance. By means of distillation [9], entanglement can be improved at the expense of using several copies. In combination with teleportation, this method allows for the construction of quantum repeaters [10], which enable the distribution of high-quality entanglement for long distance quantum communication with a favorable scaling of resources. Unfortunately, existing schemes for distillation and teleportation are incompatible with protocols generating entanglement in a steady state, since they require the decoupling of the system from the environment, such that the advantages are lost. Hence, new procedures which are suitable to accommodate dissipative methods such that all advantages can be retained and used for quantum repeaters are highly desirable.

We introduce and analyze different dissipatively driven distillation protocols, which allow for the production of highly entangled steady states independent of the initial one and present a novel quantum repeater scheme featuring the same properties. More specifically, this protocol continuously produces high-quality long-range entanglement. The required resources scale only polynomially in the distance. Once the system is operating in steady state, the resulting entangled link can be used for applications. Remarkably, the time required to drive a new pair into a highly entangled steady state is independent of the length of the link such that this setup provides a continuous supply of long distance entanglement [10]. Apart from that, the proposed distillation protocols exhibit several intriguing features. We present, for example, a method which allows for distillation in steady state where none of the individual source pairs is entangled, and describe another one whose performance can be improved by deliberately adding noise to the system.

In the following, we introduce two types of dissipative distillation protocols suitable for different situations. We start out by explaining scheme I which is physically motivated and consider the situation shown in Fig. 1. Two parties, Alice and Bob, share two source qubit pairs s_1 and s_2 , which are each dissipatively driven into an entangled steady state and used as resource for creating a single highly entangled pair in target system \mathcal{T} . As-

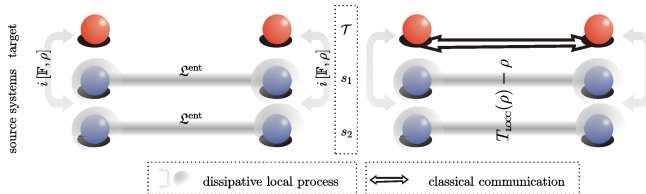


FIG. 1: (Color online) Entanglement distillation by dissipation a) Distillation setup without communication. b) Distillation setup including classical communication.

suming Markov dynamics, the time evolution of the density matrix ρ can be described by a master equation of Lindblad form $\dot{\rho} = \gamma (A\rho A^\dagger - \frac{1}{2}(\rho A^\dagger A + A^\dagger A\rho))$ with rate γ and will be abbreviated by the short hand notation $\dot{\rho} = \gamma \mathcal{L}^A(\rho)$ in the following. The entangling dissipative process acting on the source qubits considered here, is the single-particle version of the collective dynamics realized in [8] and corresponds to the master equation $\dot{\rho} = \mathcal{L}^{\text{ent}}(\rho) = \gamma (\mathcal{L}^A(\rho) + \mathcal{L}^B(\rho))$ with $A = \cosh(r)\sigma_{\text{Alice}}^- + \sinh(r)\sigma_{\text{Bob}}^+$ and $B = \cosh(r)\sigma_{\text{Bob}}^- + \sinh(r)\sigma_{\text{Alice}}^+$, where $\sigma^- = |0\rangle\langle 1|$ and $\sigma^+ = |1\rangle\langle 0|$. The unique steady state of this evolution is the pure entangled state $|\psi\rangle = (|00\rangle - \lambda|11\rangle)/\sqrt{1+\lambda^2}$, where $\lambda = \tanh(r)$. It is subject to local cooling, heating and dephasing noise described by $\mathcal{L}^{\text{noise}}(\rho) = \varepsilon_c (\mathcal{L}^{\sigma_{\text{Alice}}^-}(\rho) + \mathcal{L}^{\sigma_{\text{Bob}}^-}(\rho)) + \varepsilon_h (\mathcal{L}^{\sigma_{\text{Alice}}^+}(\rho) + \mathcal{L}^{\sigma_{\text{Bob}}^+}(\rho)) + \varepsilon_d (\mathcal{L}^{|1\rangle\langle 1|_{\text{Alice}}}(\rho) + \mathcal{L}^{|1\rangle\langle 1|_{\text{Bob}}}(\rho))$. We assume that the entangling dynamics acting on s_1 and s_2 is noisy, while the target system is protected (this assumption will be lifted below). The source qubits are locally coupled to \mathcal{T} such that

$$\dot{\rho} = \mathcal{L}_{s_1}^{\text{ent}}(\rho) + \mathcal{L}_{s_2}^{\text{ent}}(\rho) + \mathcal{L}_{s_1}^{\text{noise}}(\rho) + \mathcal{L}_{s_2}^{\text{noise}}(\rho) + \mathcal{L}_{\text{Alice}}(\rho) + \mathcal{L}_{\text{Bob}}(\rho),$$

where $\mathcal{L}_{\text{Alice}}(\mathcal{L}_{\text{Bob}})$ acts only on Alice's (Bob's) side. We choose $\mathcal{L}_{\text{Alice}}(\rho) = -\mathcal{L}_{\text{Bob}}(\rho) = i\delta_{\mathbb{F}}[\mathbb{F}, \rho]$, corresponding to the unitary evolution with respect to the Hamiltonian $\mathbb{F} = \sum_{i,j} |j_t i_s\rangle\langle i_t j_s|$, where $|\hat{0}_s\rangle = |0_{s_1} 1_{s_2}\rangle$ and $|\hat{1}_s\rangle = |1_{s_1} 0_{s_2}\rangle$. Note that this distillation protocol does not require any classical communication or pre-defined correlations. As can be seen in Fig. 2a, the efficiency is mainly determined by the mixedness of the source states rather than their entanglement. In the absence of errors, the target system reaches a maximally entangled state.

In order to allow also for noise acting on \mathcal{T} , we include now classical communication. As shown in the appendix, any Lindblad operator of the form $\mathcal{L}^{T_{\text{LOCC}}}(\rho) = (T_{\text{LOCC}}(\rho) - \rho)$, where T_{LOCC} is an arbitrary LOCC channel [11], can be realized using local dissipative processes in combination with classical communication [12]. In particular, this allows for the stabilization of the distillation schemes discussed below against errors acting on the target system by running them using m blocks of source pairs, which are all coupled to the same target state (see Sec. 3 in [13]) as illustrated in Fig. 3a. If sufficiently many source-blocks, m , are used, the dynamics is dominated by the desired processes. For clarity, we discuss the following distillation schemes in the absence of target errors, which corresponds exactly to the limit $m \rightarrow \infty$.

Thus, we consider the master equation

$$\dot{\rho} = \mathcal{L}_{s_1}^{\text{ent}}(\rho) + \mathcal{L}_{s_2}^{\text{ent}}(\rho) + \mathcal{L}_{s_1}^{\text{noise}}(\rho) + \mathcal{L}_{s_2}^{\text{noise}}(\rho) + \delta_{\mathbb{F}}(T_{\mathbb{F}}(\rho) - \rho).$$

The LOCC map $T_{\mathbb{F}}(\rho)$ is defined by the four Kraus operators $\mathbb{F}_A \otimes \mathbb{F}_B, P_A^\perp \otimes P_B, P_A \otimes P_B^\perp, P_A^\perp \otimes P_B^\perp$, where P, P^\perp are the projection onto the one excitation subspace and its orthogonal complement. Alice and Bob measure the

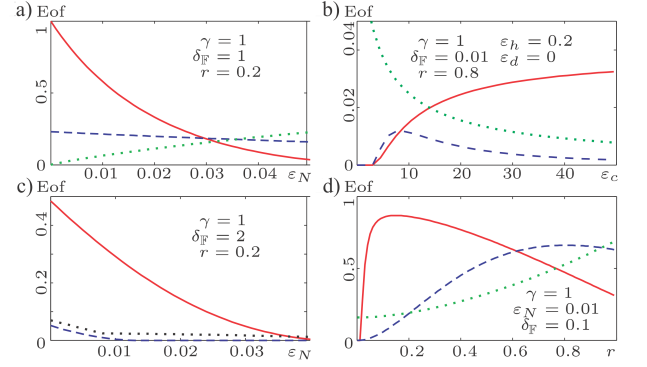


FIG. 2: (Color online) Dissipative distillation according to scheme I without communication (panel a) and including classical communication (panels b-d). The full red lines show the steady state entanglement of formation (Eof) of system \mathcal{T} . The dashed blue lines depict the steady state Eof of the source state s_1 if no distillation is performed (a,c,d) and during the protocol (b). For better visibility the blue dashed curve is multiplied by a factor 30 in panels b and c. The dotted green lines show the entropy of s_1 which is a measure of its mixedness. a) Eof attainable without communication versus error rate $\varepsilon_N \equiv \varepsilon_h = \varepsilon_c = \varepsilon_d$. b) Eof versus the noise parameter ε_c . c) Eof versus error rate ε_N . The black dotted curve represents the entanglement of the total source system measured in log negativity. d) Eof versus the parameter r .

number of excitations on their side. After successful projection onto the subspace with one excitation $P_A \otimes P_B$, a flip operation \mathbb{F} is performed, in the unsuccessful case no operation is carried out.

As shown in Fig. 2b, the scheme is robust against local noise of cooling-type ($\mathcal{L}^{\sigma}(\rho)$). This kind of noise can even be used to enhance the performance of the distillation protocol in the steady state at the cost of a lower convergence rate. Thus, counterintuitively, it can be beneficial to add noise to the system in order to increase the distilled entanglement. Moreover, the steady state entanglement of the source pairs is zero in the absence of cooling noise for the parameters considered in Fig. 2b, if no distillation scheme is performed. For increasing ε_c , the entanglement in s_1 and s_2 increases, reaches an optimal point and decreases again. Yet, the entanglement that can be distilled from these pairs is monotonously increasing and displays a boost effect. Panel c also hints at another counterintuitive effect, namely that entanglement can be distilled even though none of the source pairs is (individually) entangled in the steady state. This can be explained by noticing that the two-copy entanglement can be maintained for high noise rates when the single-copy entanglement is already vanishing. Fig. 2d shows that the distilled entanglement increases considerably for small values of the parameter r despite the decrease in the entanglement of the source pairs. This is due to the fact that the protocol is most efficient for source states

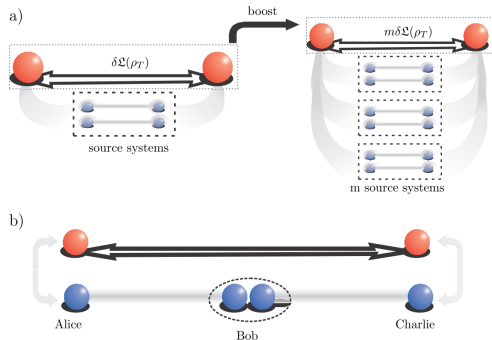


FIG. 3: (Color online) Building blocks of a dissipative quantum repeater. a) Noise resistant distillation setup. The process acting on the target system is boosted using several copies of the source system. b) Continuous entanglement swapping procedure.

close to pure states, where it allows one to distill quickly highly entangled state.

In settings where the source states can be highly mixed, another distillation scheme (scheme II hereafter) is a method of choice and will be explained in the following. We analyze a generic model, which can be solved exactly and allows one to reduce the discussion to the essential features of dissipative entanglement distillation. As in standard distillation schemes, we study the general problem in terms of Werner states [14], since many situations can be described this way and a wide range of processes can be cast in this form by twirling [14]. Werner states are of the simple form $\rho_W(f) = f\Omega + (1-f)(\mathbb{I} - \Omega)/3$, where Ω is a projector onto the maximally entangled state $|00\rangle + |11\rangle$, and \mathbb{I} the identity operator. We assume a process which drives each source pair into the state Ω , $\dot{\rho} = \gamma(\text{tr}(\rho)\Omega - \rho) \equiv \gamma E(\rho)$. Local depolarizing noise is added in the form of the Lindblad term $N(\rho) \equiv (\rho_{\text{Alice}} \otimes \mathbb{I} - \rho) + (\mathbb{I} \otimes \rho_{\text{Bob}} - \rho)$, where ρ_{Alice} (ρ_{Bob}) denotes the reduced density matrix of Alice's (Bob's) system and \mathbb{I} the normalized identity. This term describes the continuous replacement of the initial state by the completely mixed one. The source system reaches the steady state $\rho_s \propto \gamma\Omega + \varepsilon\mathbb{I}$ of the total master equation $\dot{\rho} = \gamma E(\rho) + \frac{\varepsilon}{2}N(\rho)$ at least exponentially fast in γ (see Sec. 4 in [13]). A continuous distillation process based on a standard protocol [15] can be constructed considering n source pairs which are independently driven into the steady state ρ_s and a target system \mathcal{T} . \mathcal{T} is coupled to the source pairs by a dissipative dynamics of the form $\dot{\rho} = \delta_D(\text{tr}(\rho)T_D(\rho) - \rho)$, where the completely positive map $T_D(\rho)$ corresponds to a process which acts on the n source pairs and distills a single potentially higher entangled copy. The output state is written on \mathcal{T} , while the n source pairs are re-initialized in the state \mathbb{I} . The total

master equation is given by

$$\dot{\rho} = \sum_{i=1}^n \left(\gamma E_i(\rho) + \frac{\varepsilon}{2} N_i(\rho) \right) + \delta_D(T_D(\rho) - \rho),$$

where E_i , N_i denote entangling and noise processes on the i th source qubit pair. The steady state has a fidelity of $f = \int_0^1 dx f_D(f_s - (f_s - 0.25)x^{\frac{\gamma+\varepsilon}{\delta_D}})$, where f_s and $f_D(f) = \text{tr}(\Omega T_D(\rho_W(f)^{\otimes n}))$ are the fidelity of ρ_s and the output of the distillation protocol with n input states of fidelity f . High fidelities require low values of δ_D . However, the solution $\rho(t)$ (see [13], Sec. 4) shows that fast convergence requires high values of this parameter. A low convergence speed on the target system is extremely disadvantageous if noise is acting on \mathcal{T} . Therefore, a boost of the process as illustrated in Fig. 3a is required. This way, the new convergence rate is given by $m\delta_D$ while the back action on each source system remains unchanged (see [13], Sec. 3).

The distribution of entanglement over large distances is one of the big challenges in quantum information science. In quantum repeater schemes, entanglement is generated over short distances with high accuracy and neighboring links are connected by entanglement swapping. This procedure allows one to double the length of the links, but comes at the cost of a decrease in entanglement for non-maximally entangled states. Therefore a distillation scheme has to be applied before proceeding to the next stage, which consists again of entanglement swapping and subsequent distillation. The basic setup for a continuous entanglement swapping procedure is sketched in Fig. 3b. It consists of three nodes operated by Alice, Bob and Charlie, where Alice and Bob as well as Bob and Charlie share an entangled steady state. By performing a teleportation procedure, an entangled link is established between Alice and Charlie and written onto the target system, while the source systems are re-initialized in the state \mathbb{I} . This corresponds to LOCC operation $T_{\text{sw}}(\rho)$. The whole dynamics is described by the master equation

$$\dot{\rho} = \sum_{i=1}^2 \left(\gamma E_i(\rho) + \frac{\varepsilon}{2} N_i(\rho) \right) + \delta_{\text{sw}}(T_{\text{sw}}(\rho) - \rho).$$

The steady state has a target fidelity of $f = \frac{2\gamma^2}{(2\gamma + \delta_{\text{sw}})(\gamma + \delta_{\text{sw}})}(f_{\text{sw}}(f_s) - \frac{1}{4}) + \frac{1}{4}$, where $f_{\text{sw}}(f_s)$ is the output fidelity of the entanglement swapping protocol for two input states with fidelity f_s (see [13], Sec. 5).

The basic idea of a nested steady state quantum repeater is illustrated in Fig. 4. At the lowest level, entangled steady states are generated over a distance L_0 . At each new level, two neighboring states are connected via a continuous entanglement swapping procedure and subsequently written onto a target pair separated by twice the distance. The distillation and boost processes, that are required in each level to keep the fidelity constant are not shown in this picture. The resources required for this

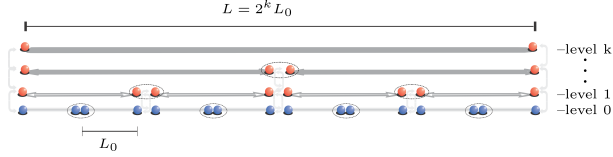


FIG. 4: (Color online) Steady state quantum repeater scheme.

repeater scheme can be estimated as follows. Entanglement swapping processes acting on source pairs of length l with fidelity f_l result in entangled target pairs of length $2l$, with degraded fidelity $f_{2l} < f_l$. This reduction is due to the swapping procedure, noise acting on the target system and the back-action from entanglement distillation. Stabilization against noise acting on the target systems is achieved by coupling each of them to m copies of the source system and requires therefore $2m$ source pairs of length l . In order to obtain a fidelity $f_{2l} \geq f_l$, n copies of these error stabilized links are used as input for a n to 1 distillation process. The distilled state is mapped to another target pair of length $2l$, which also needs to be stabilized against errors using m copies of the blocks described. Hence, in total $2m^2n$ pairs of length l are required for a repeater stage which doubles the distance over which entanglement is distributed. For creating a link of length $L = L_0 2^k$, $(2m^2n)^k$ source pairs are needed, where k is the number of required iterations of the repeater protocol. Therefore, the required resources scale polynomial with $(L/L_0)^{\log_2(2m^2n)}$. In Sec. 5 in [13], we discuss a specific example scaling with $(L/L_0)^{16.4}$. The convergence time of the total system scales only logarithmically with the distance L . Once the steady state is reached, the entanglement of the last target system can be used e.g. for quantum communication or cryptography. The underlying source systems are not effected by this process and remain in the steady state. Therefore, the target state is restored in constant time.

In conclusion, we have shown how entanglement can be distilled in a steady state and distributed over long distances by means of a dissipative quantum repeater scheme serving as stepping stone for future work aiming at the optimization in view of efficiency and experimental implementations.

APPENDIX The continuous exchange of classical communication is added in the framework of dissipative quantum information processing, by assuming that Alice and Bob have access to a system, which is used for communication only and considering the master equation

$$\dot{\rho} = \Gamma \left(\sum_i \langle i_{c_A} | \rho_{\text{Alice}} | i_{c_A} \rangle | 0_{c_A} i_{c_B} \rangle \langle 0_{c_A} i_{c_B} | - \rho \right) \equiv \Gamma \mathfrak{C}_{A \rightarrow B}(\rho)$$

States referring to the communication system at Alice's and Bob's side are labelled by subscripts c_A and c_B . Alice's communication system is continuously measured

in the computational basis yielding the quantum state $|i_{c_A}\rangle$ with probability $\langle i_{c_A} | \rho_{\text{Alice}} | i_{c_A} \rangle$ and reset to the state $|0_{c_A}\rangle$, while the communication system on Bob's side is set to the measurement outcome. This way, classical information can be sent at a rate Γ , but no entanglement can be created (see [13], Sec. 2). As proven in Sec. 2 in [13], any operation that can be realized by means of local operations and classical communication (LOCC) can be constructed in a continuous fashion using communication processes $\mathfrak{C}_{A \rightarrow B}$ and $\mathfrak{C}_{B \rightarrow A}$, if the rate Γ is fast compared to all other relevant processes including the retardation due to back and forth communication.

We thank Eugene Polzik for helpful discussions and acknowledge support from the Elite Network of Bavaria (ENB) project QCCC, the DFG-Forschungsgruppe 635 and the EU projects COMPAS and QUEVADIS.

-
- [1] M. B. Plenio and S. F. Huelga, Phys. Rev. Lett. **88**, 197901(2002); B. Kraus and J. I. Cirac, Phys. Rev. Lett. **92**, 013602 (2004); F. Benatti, R. Floreanini, and U. Marzolino, Phys. Rev. A **81**, 012105 (2010); R. Bloomer, M. Pysher and O. Pfister, arXiv:1007.2369 (2010); J. Cho, S. Bose and M. S. Kim, Phys. Rev. Lett. **106**, 020504 (2011) (2010); M. J. Kastoryano, F. Reiter, and A. S. Sørensen, Phys. Rev. Lett. **106**, 090502 (2011); A. Mari, J. Eisert, arXiv:1104.0260 (2011); J. T. Barreiro et al., Nature **470**, 486 (2011).
 - [2] J. F. Poyatos, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **77**, 4728 (1996).
 - [3] S. Diehl et al. Nature Phys. **4**, 878 (2008).
 - [4] F. Verstraete, M. M. Wolf, and J. I. Cirac, Nature Phys. **5**, 633 (2009).
 - [5] F. Pastawski, L. Clemente, and J. I. Cirac, Phys. Rev. A **83**, 012304 (2011).
 - [6] C. A. Muschik, E. S. Polzik, and J. I. Cirac, arXiv:1007.2209 (2010), see also: A. S. Parkins, E. Solano, and J. I. Cirac, Phys. Rev. Lett. **96**, 053602 (2006).
 - [7] Joseph Kerckhoff, Hendra I. Nurdin, Dmitri S. Pavlichin, Hideo Mabuchi, Phys. Rev. Lett. **105**, 040502 (2010); J. P. Paz and W. H. Zurek, Proc. of the Royal Soc. of London. Series A **454**, 355 (Jan. 1998).
 - [8] H. Krauter et al. arXiv:1006.4344 (2010).
 - [9] C. H. Bennett et al. Phys. Rev. Lett. **76** 722 (1996).
 - [10] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **81**, 5932 (1998).
 - [11] LOCC channels are completely positive trace preserving maps that can be realized by means of Local Operations and Classical Communication.
 - [12] We assume, that the time scales for classical communication Γ^{-1} (see appendix) are sufficiently long such that retardation effects can be ignored.
 - [13] Supplemental Material
 - [14] R. F. Werner, Phys. Rev. A **40**, 4277 (1989); M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
 - [15] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).

SUPPLEMENTAL MATERIAL

Here, we explain the results presented in the main text. In Sec. 1 and Sec. 2, two dissipative distillation schemes are discussed. Scheme I is suited for settings where a dissipative process is available which produces entangled steady states, that are close to pure states. If only very mixed steady states are available as input, scheme II is preferable (which we explain in detail in Sec. 4). In Sec. 2, the notion of continuous exchange of classical information between two parties is introduced in the master equation formalism and it is shown that arbitrary LOCC channels can be realized using local dissipation and classical communication. In Sec. 3, we explain how the continuous protocols used here can be made robust against noise. Finally, in Sec. 5, we analyze the dissipative quantum repeater scheme put forward in the main text in detail.

1. SCHEME I: DISSIPATIVE ENTANGLEMENT DISTILLATION FOR SOURCE STATES CLOSE TO PURE STATES

In this section, we explain two variants of scheme I [S1]. In Sec. 1.1, we discuss a protocol, which allows for dissipative entanglement distillation without communication. Sec. 1.2 is concerned with a related protocol, which includes classical communication. Both protocols produce Bell-diagonal steady states which can further be distilled using scheme II presented in Sec. 4.

1.1 Dissipative entanglement distillation without communication

We consider the setup illustrated in Fig. S.1. The dissipative dynamics driving the two systems s_1 and s_2 is physically motivated and can be implemented by coupling the systems located at Alice's and Bob's side to a common bath, for example the vacuum modes of the electromagnetic field [6, 8]. The entanglement which can be attained per single copy is limited for a given dissipative process. Moreover these systems are subject to noise. Still, it is possible to use these two copies as resource for creating a single highly entangled pair in target system \mathcal{T} . In the absence of undesired processes, the dynamics described by the master equation $\dot{\rho} = \gamma (\mathfrak{L}^A(\rho) + \mathfrak{L}^B(\rho))$ (see main text) drives systems s_1 and s_2 into the state

$$|\psi\rangle^{\otimes 2} \propto |00\rangle_{s_1}|00\rangle_{s_2} - \lambda[|00\rangle_{s_1}|11\rangle_{s_2} + |11\rangle_{s_1}|00\rangle_{s_2}] + \lambda^2|11\rangle_{s_1}|11\rangle_{s_2}.$$

Alice and Bob share a maximally entangled state $|\Psi_0\rangle = (|00\rangle_{s_1}|11\rangle_{s_2} + |11\rangle_{s_1}|00\rangle_{s_2})/\sqrt{2}$ in a subspace with one excitation on each side. Scheme I is based on the extraction of entanglement from this subspace and its subsequent transfer to the target system by means of the flip operation $\mathbb{F} = \sum_{i,j} |j\mathcal{T}\hat{i}_s\rangle\langle i\mathcal{T}\hat{j}_s|$, where $|\hat{0}_s\rangle = |0_{s_1}1_{s_2}\rangle$ and

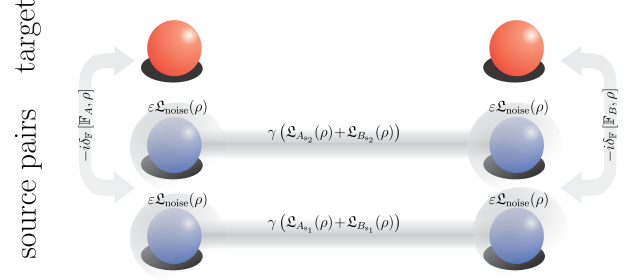


FIG. S.1: (Color online) Scheme I, dissipative entanglement distillation without communication.

$|\hat{1}_s\rangle = |1_{s_1}0_{s_2}\rangle$. Systems s_1 and s_2 are permanently driven back to an entangled state. In contrast to standard distillation protocols for pure states [S10], the presence of this strong process leads to a substantial decrease in the entanglement if the flip operations on Alice's and Bob's side are not applied simultaneously. Hence, the coordination of their actions, e.g., using fast classical communication, seems to be essential. Surprisingly, the desired dynamics can be realized in the absence of communication or predefined correlations using local unitary evolutions.

This is possible by exploiting the symmetry of the maximally entangled state $|\Psi_0\rangle$. More specifically, $|\Psi_0\rangle$ is invariant under any unitary operation of the form $U \otimes \bar{U}$, while less entangled pure states are not. \bar{U} denotes the complex conjugate of U . Such an operation can be implemented without communication as the time evolution of a sum of local Hamiltonians $H = H_A \otimes \mathbb{I} - \mathbb{I} \otimes \bar{H}_B$. Here, we use the flip operation such that the corresponding master equation is given by

$$\begin{aligned} \dot{\rho} = & \gamma (\mathfrak{L}^{A_{s_1}}(\rho) + \mathfrak{L}^{B_{s_1}}(\rho) + \mathfrak{L}^{A_{s_2}}(\rho) + \mathfrak{L}^{B_{s_2}}(\rho)) \\ & + i\delta_{\mathbb{F}} [\mathbb{F} \otimes \mathbb{I} - \mathbb{I} \otimes \mathbb{F}, \rho] \\ & + \epsilon_c (\mathfrak{L}^{a_{s_1}}(\rho) + \mathfrak{L}^{b_{s_1}}(\rho) + \mathfrak{L}^{a_{s_2}}(\rho) + \mathfrak{L}^{b_{s_2}}(\rho)) \\ & + \epsilon_h (\mathfrak{L}^{a_{s_1}^\dagger}(\rho) + \mathfrak{L}^{b_{s_1}^\dagger}(\rho) + \mathfrak{L}^{a_{s_2}^\dagger}(\rho) + \mathfrak{L}^{b_{s_2}^\dagger}(\rho)) \\ & + \epsilon_d (\mathfrak{L}^{a_{s_1}^\dagger a_{s_1}}(\rho) + \mathfrak{L}^{b_{s_2}^\dagger b_{s_2}}(\rho)), \end{aligned}$$

where $a = \sigma_{\text{Alice}}^-$ and $b = \sigma_{\text{Bob}}^-$. The first line corresponds to the entangling dissipative process (described by nonlocal jump operators A and B) acting on the two source systems as explained in the main text. The second line describes the unitary coupling of the target system to the entangled subspace of the two source systems and the last three lines represent undesired processes. More specifically, we include dephasing at a rate ϵ_d as well as noise terms, which create (annihilate) excitations locally at the heating (cooling) rate ϵ_h (ϵ_c). Note that the noise types considered here also include depolarizing noise. The target system itself is assumed to be protected (below, a variant of this scheme is described, which includes classical communication and

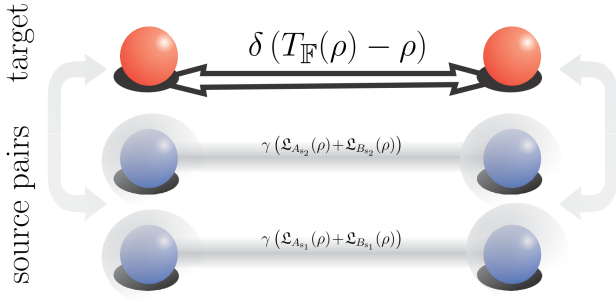


FIG. S.2: (Color online) Scheme I, dissipative entanglement distillation including classical communication channels.

can be made robust against noise acting on the target system).

A disadvantage of the unitary evolution employed here lies in the fact that the source system is subject to a back-action of the target state, which depends on the quantum state of \mathcal{T} . Accordingly, the evolution of the source systems is highly dependent on the state of the target pair. It remains an open question, whether schemes, similar to the one described in Sec. 3 can be used to render this protocol robust against errors on the target system, or whether this is a special feature of protocols including classical communication.

1.2 Distillation using scheme I including classical communication

We consider the setup illustrated in Fig. S.2. As explained in Sec. 1.1, the dissipative entangling process acting on the source systems s_1 and s_2 has the property that Alice and Bob share a maximally entangled state if the resulting steady state is projected onto the subspace with one excitation on each side. Ideally, this quantum state is then transferred to the target system \mathcal{T} by means of the flip operation defined above. In this subsection, we introduce a classical communication channel, which allows Alice and Bob to coordinate their actions such that flip operations on both sides can be performed in a synchronized fashion if both sides have successfully accomplished a projection onto the relevant subspace with one excitation. As explained in Sec. 2, Lindblad terms of the form $\mathfrak{L}^{T_{\text{LOCC}}}(\rho) = (T_{\text{LOCC}}(\rho) - \rho)$, where T_{LOCC} is an arbitrary LOCC channel [11], can be realized by means of local dissipative processes and classical communication.

As explained in Sec. 3, this protocol is resistant against target errors if it is coupled to sufficiently many blocks source pairs. For simplicity, we explain here the basic protocol in the absence of target errors, which corresponds to the limit of using infinitely many source blocks (entanglement distillation for a finite number of source

blocks and finite error rates is analyzed in Sec. 3 and Sec. 5). Classical communication allows for the implementation of the scheme outlined above. The LOCC distillation operation corresponding to this process, $T_{\mathbb{F}}$, is given by

$$\begin{aligned} T_{\mathbb{F}}(\rho) = & \mathbb{F}_A \otimes \mathbb{F}_B \rho \mathbb{F}_A \otimes \mathbb{F}_B \\ & + P_A \otimes P_B^\perp \rho P_A \otimes P_B^\perp \\ & + P_A^\perp \otimes P_B \rho P_A^\perp \otimes P_B \\ & + P_A^\perp \otimes P_B^\perp \rho P_A^\perp \otimes P_B^\perp, \end{aligned}$$

where $P = |0_{s_1}1_{s_2}\rangle\langle 0_{s_1}1_{s_2}| + |1_{s_1}0_{s_2}\rangle\langle 1_{s_1}0_{s_2}|$ is the projector onto the subspace with one excitation, and $P^\perp = \mathbb{I} - P$ the projector onto the subspace with zero or two excitations. Note that only the first term has an effect on the target system. The flip operation leads to a back-action on the source system, which depends on the state of \mathcal{T} . In order to simplify the discussion in Sec. 3, we introduce here a slightly modified version of this protocol, $T'_{\mathbb{F}}(\rho)$, which does not exhibit a state-dependent back-action. This can be avoided by applying a twirl [14] on the target system prior to the flip operation

$$\begin{aligned} T'_{\mathbb{F}}(\rho) = & \sum_{ij=0}^3 \frac{1}{16} \mathbb{F}_A \otimes \mathbb{F}_B U_{ij} \rho U_{ij}^\dagger \mathbb{F}_A \otimes \mathbb{F}_B \\ & + P_A \otimes P_B^\perp \rho P_A \otimes P_B^\perp \\ & + P_A^\perp \otimes P_B \rho P_A^\perp \otimes P_B \\ & + P_A^\perp \otimes P_B^\perp \rho P_A^\perp \otimes P_B^\perp. \end{aligned}$$

$U_{ij} = \sigma_i^A \otimes \sigma_j^B$ is a unitary operation acting on the target system only, where σ_i denote the four Pauli matrices and σ_0 is the identity. Due to the twirl, this protocol features an enhanced back-action on the source, which is independent from the target. It turns out that the performance of this protocol is qualitatively the same as shown in Fig. 2 in the main text. The total master equation is then given by

$$\begin{aligned} \dot{\rho} = & \gamma(\mathfrak{L}^{A_{s_1}}(\rho) + \mathfrak{L}^{B_{s_1}}(\rho) + \mathfrak{L}^{A_{s_2}}(\rho) + \mathfrak{L}^{B_{s_2}}(\rho)) \\ & + \delta_{\mathbb{F}}(T'_{\mathbb{F}}(\rho) - \rho) \\ & + \varepsilon_c(\mathfrak{L}^{a_{s_1}}(\rho) + \mathfrak{L}^{b_{s_1}}(\rho) + \mathfrak{L}^{a_{s_2}}(\rho) + \mathfrak{L}^{b_{s_2}}(\rho)) \\ & + \varepsilon_h(\mathfrak{L}^{a_{s_1}^\dagger}(\rho) + \mathfrak{L}^{b_{s_1}^\dagger}(\rho) + \mathfrak{L}^{a_{s_2}^\dagger}(\rho) + \mathfrak{L}^{b_{s_2}^\dagger}(\rho)) \\ & + \varepsilon_d(\mathfrak{L}^{a_{s_1}^\dagger a_{s_1}}(\rho) + \mathfrak{L}^{b_{s_2}^\dagger b_{s_2}}(\rho)), \end{aligned}$$

where $a = \sigma_{\text{Alice}}^-$ and $b = \sigma_{\text{Bob}}^-$.

2. CLASSICAL DISSIPATIVE CHANNELS AND DISSIPATIVE LOCC

Classical channels are easier to realize experimentally than their quantum counterparts and can for example be



FIG. S.3: (Color online) Realization of a classical dissipative channel.

implemented using optical fibers. Since classical channels are insufficient for the generation of quantum correlations, long-range links can be established over large distances using the toolkit of classical error-correction. The class of LOCC operations, i.e. quantum operations that can be performed using local operations and classical communication, is of essential importance in quantum information theory, especially in the context of entanglement distillation protocols.

In this section, we introduce the notion of classical channels in the framework of dissipative quantum information processing. This allows us to formulate generalized LOCC operations in a continuous dissipative setting, which includes a wide range of continuous distillation protocols.

2.1 Classical dissipative channels

We start out by introducing a dissipative classical communication channel. Both parties, Alice and Bob, each have access to a d -dimensional system which is used exclusively for classical communication (see. Fig. S.3). The master equation

$$\dot{\rho} = \Gamma \left(\sum_i \langle i_{c_A} | \rho_{\text{Alice}} | i_{c_A} \rangle | 0_{c_A} i_{c_B} \rangle \langle 0_{c_A} i_{c_B} | - \rho \right) \equiv \Gamma \mathcal{C}_{A \rightarrow B}(\rho).$$

describes a one-way classical communication channel. States referring to the communication system at Alice's and Bob's side are labelled by subscripts c_A and c_B respectively. Alice's communication system is continuously measured in the computational basis yielding the quantum state $|i_{c_A}\rangle$ with probability $\langle i_{c_A} | \rho_{\text{Alice}} | i_{c_A} \rangle$ and reset to the state $|0_{c_A}\rangle$, while the communication system on Bob's side is set to the measurement outcome. This process can be written in the form

$$\dot{\rho} = \Gamma \mathcal{C}_{A \rightarrow B}(\rho) \equiv \Gamma(T(\rho) - \rho),$$

where the completely positive map $T(\rho)$ is an entanglement breaking operation [S2], which maps any state to a separable one. The solution of this master equation $\rho(t)$ is given by

$$\rho(t) = \rho(0)e^{-\Gamma t} + \underbrace{\int_0^t d\tau T(\rho(\tau))e^{\Gamma(\tau-t)}}_{\text{separable}}.$$

The second term is separable, since $T(\rho)$ is entanglement breaking. Accordingly, the classical channel introduced above does not produce entanglement. Moreover any entanglement present in the state $\rho(0)$ is exponentially suppressed.

2.2 Generation of Lindblad operators of the form $T(\rho) - \rho$

In the following we prove that any dissipative time evolution which satisfies a master equation of the form $\dot{\rho} = \gamma(T(\rho) - \rho)$ can be designed by means of local dissipative processes in combination with the classical communication channels introduced above in the limit of high rates Γ . The basic setup is sketched in Fig. S.4. Alice and Bob hold a bipartite system, which we refer to as the main system. In addition both parties have access to several classical communication channels and can apply dissipative dynamics acting on the classical channels and their part of the main system. This setting allows for a wider class of dissipative evolutions on the main system which includes dissipative LOCC processes. In particular we state the following:

Let $T(\rho)$ be any LOCC map. Let $\mathcal{L}(\rho)$ be any bounded Lindblad operator, i.e., $\max_{\rho} \|\mathcal{L}(\rho)\| = 1$, acting on the main system at a rate γ . Let Alice and Bob have access to classical communication channels as described above. If both parties can apply any dissipative process of Lindblad form on their side, an effective dissipative time evolution on the main system satisfying the master equation

$$\dot{\rho} = \gamma \mathcal{L}(\rho) + \delta (T'(\rho) - \rho) \quad (1)$$

after an initial waiting time of the order $\frac{1}{\delta}$ can be realized. The completely positive operator $T'(\rho) = T(\rho) + \mathcal{O}(\sqrt{\alpha})$ is an imperfect realization of $T(\rho)$ up to an error $\mathcal{O}(\sqrt{\alpha})$, which vanishes for small $\alpha = \frac{\gamma'}{\Gamma}$, where $\gamma' = \gamma + \delta$. $\mathcal{O}(f(\alpha))$ denotes any hermitian (time and state dependent) operator with a trace norm scaling with $f(\alpha)$ in the limit $\alpha \rightarrow 0$.

Since $\max_{\rho} \|\mathcal{L}(\rho)\| = 1$, the strength of the process is completely encoded in γ . $\mathcal{L}(\rho)$ can include a dissipative LOCC map itself, as discussed at the end of this section. The error $\mathcal{O}(\sqrt{\alpha})$ of the LOCC map is small for $\alpha \ll 1$. A time evolution satisfying Eq. (1) can be either obtained by starting from certain initial conditions, or after an initial waiting time on the order of $\frac{1}{\delta}$, during which no external control is required. If $\Gamma \gg \gamma + \delta$ ($\alpha \ll 1$), the system evolves approximately according to $\dot{\rho} = \gamma \mathcal{L}(\rho) + \delta (T(\rho) - \rho)$.

Note, that LOCC operations are extremely hard to parameterize. It is known, that they can be written as a separable superoperator $T(\rho) = \sum_i A_i \otimes B_i \rho A_i^\dagger \otimes B_i^\dagger$, but not every separable superoperator is a LOCC map. Practically, a general LOCC map can only be characterized by fixing the number of communication rounds

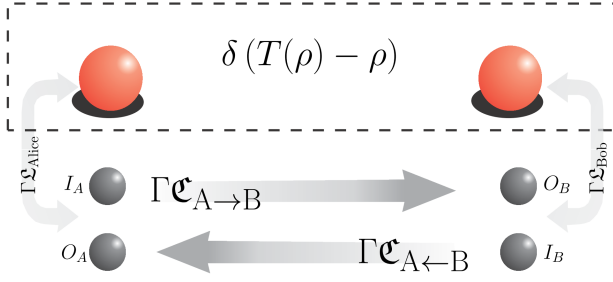


FIG. S.4: (Color online) Using local dissipative processes and fast classical communication, arbitrary LOCC channels can be implemented in a continuous fashion.

between Alice and Bob and to specifying the exact operations that Alice and Bob perform in each round. The most general operation Alice and Bob can apply is a positive operator valued measurement (POVM). This covers any completely positive map as well as measurements, unitary evolutions, etc. A POVM is specified by a number of Kraus operators A_i , corresponding to the possible measurement outcomes i , where the normalization condition $\sum_i A_i^\dagger A_i = \mathbb{I}$ guarantees that the probabilities for the different possible outcomes add up to 1.

We consider the following situation. Alice performs a first POVM A_i and sends her result i to Bob. Bob chooses a POVM B_j^i depending on Alice's result i . Subsequently, he sends the result j to Alice, who chooses her next POVM A_k^{ij} which may depend on all previous measurement result. This procedure can be repeated many times. This corresponds to the application of the operation

$$T(\rho) = \sum_{i_0, j_0, i_1, j_1, \dots, i_r, j_r} X_{i_0, j_0, i_1, j_1, \dots, i_r, j_r} \rho X_{i_0, j_0, i_1, j_1, \dots, i_r, j_r}^\dagger,$$

where each Kraus operator is of the form

$$X_{i_0, j_0, i_1, j_1, \dots, i_r, j_r} = B_{j_n}^{i_0, j_0, i_1 \dots i_n} \dots B_{j_1}^{i_0, j_0, i_1} A_{i_1}^{i_0, j_0} B_{j_0}^{i_0} A_{i_0}$$

and represent one possible set of measurements outcomes $i_0, j_0, i_1, j_1, \dots, i_r, j_r$ for all r POVM measurements. Due to this lack of a concise notation for a general LOCC map, a complete proof of this statement would be lost in notation and it would be hard for the reader to understand the main idea of the proof. We restrict ourselves therefore to LOCC maps with one communication round, i.e., Alice sends one message to Bob and Bob can send an answer back to Alice once. A generalization of the following proof to a LOCC map with a finite number of communication rounds m is straight forward and will be discussed below.

Let $T(\rho)$ denote a LOCC map with one round of communication. This map can be realized in the following way:

- Alice applies a POVM measurement with Kraus operators A_i , obtains the measurement result i and sends it to Bob.
- Bob performs a POVM measurement B_j^i , which can depend on i , and sends the result j to Alice. Since we assume that Alice is memoryless, Bob also sends the measurement outcome of Alice's measurement i .
- In the last step, Alice can apply any completely positive map T_{ij} on her side. This map can depend on both, i and j .

Note, that $T_{ij}(\rho) = \sum_k C_k^{ij} \rho (C_k^{ij})^\dagger$ is also a POVM map with Kraus operators C_k^{ij} , where the measurement results k are not used. Let Alice and Bob have n different measurement results for each POVM, where n can be upper bounded by the square of the dimension of the system. For typical distillation protocols on qubits, $n = 2$. Note that all indices for Kraus operators run from 1 to n , and do not start with 0. This choice allows for a shorter notation later on (the index 0 is reserved for indicating that the classical channel is operable). The basic setup is illustrated in Fig. S.4. Alice and Bob have access to classical one-way communication channels labelled $C1$ and $C2$. $C1$ and $C2$ can be used to send information from Alice to Bob and vice versa respectively. Apart from these classical channels, Alice and Bob hold a system subject to a dissipative evolution described by the Lindblad operator $\mathfrak{L}(\rho)$. In the following, this system is referred to as the main system. The first classical channel needs to store all possible measurement outcomes obtained by Alice, whereas the second one needs to store the measurement results obtained by both, Alice and Bob. We assume therefore that $C1$ and $C2$ are $n + 1$ and $n' + 1 = n^2 + 1$ dimensional systems respectively. Note, that the state $|0\rangle$ will be used to indicate that the channel input or output is "empty", while the states $|1\rangle, \dots, |n\rangle$ represent n possible measurement results of Alice and the states $|1\rangle, \dots, |n'\rangle$ encode the n^2 different measurement results obtained by Alice and Bob. The corresponding master equation is given by

$$\dot{\rho} = \gamma\mathfrak{L}(\rho) + \Gamma\mathfrak{C}_{A \rightarrow B}(\rho) + \Gamma\mathfrak{C}_{A \leftarrow B}(\rho),$$

where $\gamma\mathfrak{L}(\rho)$ is a process acting on the main system only. We assume, that the time scales for classical communication Γ^{-1} are sufficiently long such that retardation effects can be ignored. The four systems used for classical communication are denoted by I_a, I_b, O_a, O_b as shown in Fig. S.4. I and O stand for "Input" and "Output".

As a next step, local Lindblad operators are added, which correspond to the application of a LOCC map depending on the registers of the classical channels. The following three terms are added, one for each step of the

protocol outlined above. The first term is given by,

$$\delta \sum_{i=1, k=0} \mathfrak{L}^{A_i \otimes |i\rangle\langle k|_{I_a}}(\rho),$$

where A_i acts on Alice's part of the main system and $|i\rangle\langle 0|_{I_a}$ on Alice's side of the first classical system, i.e., the input system of the first classical channel. Note, that we use here the short hand notation $\mathfrak{L}^A(\rho) = \gamma(A\rho A^\dagger - \frac{1}{2}(\rho A^\dagger A + A^\dagger A\rho))$ that was already introduced in the main text. This corresponds to the first step of the realization of the LOCC map. Alice performs a POVM and writes the measurement result onto the input system of the classical channel. As second term, we add the Lindblad operator

$$\Gamma \sum_{ji=1, xy=0} \mathfrak{L}^{B_j^i \otimes |0\rangle\langle i|_{O_b} \otimes |j, i\rangle\langle x, y|_{I_b}}(\rho)$$

where B_j^i acts on Bob's part of the main system, $|0\rangle\langle i|_{O_b}$ on the output of the first classical channel and $|j, i\rangle\langle x, y|_{I_b}$ on the input of the second channel. Note, that the second channel can store both values i and j at the same time. $|j, i\rangle$ stands for any encoding of i, j in the $n^2 + 1$ dimensional state space, where the label zero is reserved for indicating the status of the channel. The summation over x, y starts from zero, i.e., includes the reserved zeros term as well as the n^2 possible measurement results. Bob only carries out a POVM measurement, if he receives the message i via $C1$. Afterwards he writes i, j onto the classical channel. Note that the sum over xy implies that Bob overwrites any previous state of the classical communication system. The last term to be added is given by

$$\Gamma \sum_{jik=1} \mathfrak{L}^{C_k^{ij} \otimes |0\rangle\langle i, j|_{O_a}}(\rho)$$

where C_k^{ij} acts on Alice's quantum system with $T_{ij}(\rho) = \sum_k C_k^{ij} \rho C_k^{ij\dagger}$ and $|0\rangle\langle i, j|_{O_a}$ act on the output of the second classical channel. Alice receives the message ij and reacts by applying T_{ij} to complete the LOCC map. The sum starts from $ij = 1$, i.e., Alice acts only if a message has arrived. She does not act if the register is empty ($|0\rangle$). Hence, the total master equation is given by

$$\begin{aligned} \dot{\rho} = & \gamma \mathfrak{L}(\rho) + \Gamma \mathfrak{C}_{A \rightarrow B} + \Gamma \mathfrak{C}_{A \leftarrow B} \\ & \delta \sum_{i=1, k=0} \mathfrak{L}^{A_i \otimes |i\rangle\langle k|_{I_a}}(\rho) \\ & + \Gamma \sum_{ji=1, xy=0} \mathfrak{L}^{B_j^i \otimes |0\rangle\langle i|_{O_b} \otimes |j, i\rangle\langle xy|_{I_b}}(\rho) \\ & + \Gamma \sum_{jik=1} \mathfrak{L}^{C_k^{ij} \otimes |0\rangle\langle i, j|_{O_a}}(\rho) \end{aligned} \quad (2)$$

The basic idea can be described as follows. The term in the second line starts the process of realizing $T(\rho)$ at a

rate δ (Alice performs the first step). The following steps are performed with a high rate Γ , such that the state of the quantum system stays approximately constant during the time needed to complete the whole operation. So, practically, the whole LOCC map $T(\rho)$ is applied at once at a rate δ .

In the following, this will be proven rigorously by considering the effective evolution of the main systems after tracing out the classical channels. The reduced state of the main system can be written as

$$\rho_M = \sum_{ijkl} \rho_{ijkl}$$

with $\rho_{ijkl} = \langle i_{I_a} j_{O_b} k_{I_b} l_{O_a} | \rho | i_{I_a} j_{O_b} k_{I_b} l_{O_a} \rangle$, where i, j (k, l) denote the computational bases for $C1$ ($C2$). The indices are arranged such that their order corresponds to the order in the communication cycle. i refers to the input of Alice's side, j to the output on Bob's side, k to the input on Bob's side and l to the output on Alice's side. A system of differential equations for all ρ_{ijkl} can be derived using $\dot{\rho}_{ijkl} = \langle i_{I_a} j_{O_b} k_{I_b} l_{O_a} | \dot{\rho} | i_{I_a} j_{O_b} k_{I_b} l_{O_a} \rangle$ and Eq. (2). The desired terms $\rho_{0000}, \rho_{i000}, \rho_{0i00}, \rho_{00(ij)0}, \rho_{000(ij)}$ evolve according to

$$\dot{\rho}_{0000} = \gamma \mathfrak{L}(\rho_{0000}) - \delta \rho_{0000} + \Gamma \sum_{xy=1} T_{ij}(\rho_{000(xy)}), \quad (3)$$

$$\begin{aligned} \dot{\rho}_{i000} = & \gamma \mathfrak{L}(\rho_{i000}) - (\Gamma + \delta) \rho_{i000} + \delta \sum_{k=0} A_i \rho_{k000} A_i^\dagger \\ & + \Gamma \sum_{xy=1} T_{xy}(\rho_{i00(xy)}), \end{aligned} \quad (4)$$

$$\begin{aligned} \dot{\rho}_{0i00} = & \gamma \mathfrak{L}(\rho_{0i00}) - (\Gamma + \delta) \rho_{0i00} + \Gamma \sum_{k=0} \rho_{ik00} \\ & + \Gamma \sum_{xy=1} T_{xy}(\rho_{0i0(xy)}), \end{aligned} \quad (5)$$

$$\begin{aligned} \dot{\rho}_{00(ij)0} = & \gamma \mathfrak{L}(\rho_{00(ij)0}) - (\Gamma + \delta) \rho_{00(ij)0} \\ & + \Gamma \sum_{xy=0} B_j^i \rho_{0i(xy)0} B_j^{i\dagger} + \Gamma \sum_{xy=0} T_{xy}(\rho_{00(ij)(xy)}), \end{aligned} \quad (6)$$

$$\dot{\rho}_{000(ij)} = \gamma \mathfrak{L}(\rho_{000(ij)}) - (\Gamma + \delta) \rho_{000(ij)} + \Gamma \sum_{xy=0} \rho_{00(ij)(xy)}. \quad (7)$$

All other terms correspond to small errors. In the first step of the proof it is shown that after an initial waiting time, only the states $\rho_{0000}, \rho_{i000}, \rho_{0i00}, \rho_{00(ij)0}, \rho_{000(ij)}$ are significantly populated, while the population of all other states is small. In the next step it is shown that $\rho_M \approx \rho_{0000}$. In the following we will use the short-hand notation $\rho_0 := \rho_{0000}$.

Bounds for occupation probabilities

Let us define the probabilities $p_{ijkl} = \text{tr}(\rho_{ijkl})$. A system of differential equations $\dot{p}_{ijkl} = \text{tr}(\dot{\rho}_{ijkl})$ for these

probabilities can be derived by from the differential equations for ρ_{ijkl} . Traceless terms such as $\gamma\mathcal{L}(\rho)$ and T_{ij} do no longer appear.

We define $p_{000X}, \dots, p_{XXXX}$ as the sum of p_{ijkl} , where all indices marked with X are summed from 1 to n, n' , in order to remove the dependence on the Kraus operators by virtue of their normalization condition ($\sum_{i=1}^n A_i^\dagger A_i = \mathbb{I}$). A successful application of the LOCC map corresponds to the series $p_{X000}, p_{0X00}, p_{00X0}, p_{000X}, p_{0000}$ (similarly to a X-excitation which created on the first index, travels to the right and disappears in the end). p_{0000} takes high values, while the other probabilities are on the order of $\frac{\delta}{\Gamma}$, which indicates that this process is fast. However, the situation considered here does not correspond to this ideal case because Alice can start a new round before the last one is finished, which gives rise to probabilities which are denoted by indices with more than one X , e.g. $p_{XX00}, p_{X0X0}, \dots$ which results in an incorrect realization of the LOCC map. Since we are not interested in the complete solution but only in upper and lower bounds, we further simplify the system by defining the probabilities $p_{0000}, p_{X\Xi\Xi\Xi}, p_{0X\Xi\Xi}, p_{00X\Xi}$ and p_{000X} , which cover all possible events. X indicates that the corresponding index is different from zero. Therefore, the summation runs from 1 to n, n' . Ξ stands for an arbitrary value, i.e., the summation starts from zero. These five quantities include also non-ideal process, with two or more X entries which correspond an errors and evolve according to

$$\begin{aligned}\dot{p}_{0000} &= -\delta p_{0000} + \Gamma p_{000X}, \\ \dot{p}_{000X} &= -(\delta + \Gamma) p_{000X} + \Gamma p_{00X\Xi}, \\ \dot{p}_{00X\Xi} &= -(\delta + \Gamma) p_{00X\Xi} + \Gamma p_{0X\Xi\Xi}, \\ \dot{p}_{0X\Xi\Xi} &= -(\delta + \Gamma) p_{0X\Xi\Xi} + \Gamma p_{X\Xi\Xi\Xi}, \\ \dot{p}_{X\Xi\Xi\Xi} &= -\Gamma p_{X\Xi\Xi\Xi} + \delta(p_{0000} + p_{000X} + p_{00X\Xi} + p_{0X\Xi\Xi}).\end{aligned}$$

The solution shows that the steady state (ss) with

$$\begin{aligned}p_{0000}^{\text{ss}} &= \frac{\Gamma^4}{(\Gamma + \delta)^4}, \quad p_{000X}^{\text{ss}} = \frac{\delta \Gamma^3}{(\Gamma + \delta)^4}, \quad p_{00X\Xi}^{\text{ss}} = \frac{\delta \Gamma^2}{(\Gamma + \delta)^3}, \\ p_{0X\Xi\Xi}^{\text{ss}} &= \frac{\delta \Gamma}{(\Gamma + \delta)^2}, \quad p_{X\Xi\Xi\Xi}^{\text{ss}} = \frac{\delta}{(\Gamma + \delta)}.\end{aligned}\quad (8)$$

is reached up to an error smaller than $\mathcal{O}(\alpha^2)$ after a time of the order of $\frac{1}{\delta}$ [S3]. In the steady state, $p_{0000}^{\text{ss}} = 1 - 4\frac{\delta}{\Gamma} + \mathcal{O}(\frac{\delta^2}{\Gamma^2})$. Next, bounds for $p_{X000}, p_{0X00}, p_{00X0}, p_{000X}$ are derived. According to Eq. (4), $\dot{p}_{X000} = -\Gamma p_{X000} + \delta p_{0000} + \Gamma p_{X00X}$. Assuming that p_{0000}^{ss} is reached after a time t' ,

$$\begin{aligned}p_{X000}(t) &= e^{-\Gamma(t-t')} p_{X000}(t') + \int_{t'}^t d\tau e^{\Gamma(\tau-t)} \left(\frac{\delta \Gamma^4}{(\Gamma + \delta)^4} + \Gamma p_{X00X} \right), \\ &= e^{-\Gamma(t-t')} p_{X000}(t') + (1 - e^{-\Gamma(t-t')}) \frac{\delta \Gamma^3}{(\Gamma + \delta)^4} + h(t, t'),\end{aligned}$$

where $h(t, t') \geq 0$ is a positive function since $p_{X00X} \geq 0$. Hence, doubling the initial waiting time guarantees

$p_{X000} \geq \frac{\delta \Gamma^3}{(\Gamma + \delta)^4}$. According to Eq. (5), $\dot{p}_{0X00} = -(\Gamma + \delta) p_{0X00} + \Gamma p_{X000} + \Gamma p_{XX00} + \Gamma p_{0X0X}$. By integration, using the bound for p_{X000} and assuming that the contributions from p_{0X0X} and p_{XX00} sum up to a positive function we conclude that $\frac{\delta \Gamma^4}{(\Gamma + \delta)^5} \leq p_{0X00}$ is fulfilled after waiting for another period on the order of $\frac{1}{\delta}$. Similarly, one obtains $\frac{\delta \Gamma^5}{(\Gamma + \delta)^6} \leq p_{00X0}$ and $\frac{\delta \Gamma^6}{(\Gamma + \delta)^7} \leq p_{000X}$. Since $p_{0000} + p_{X000} + p_{0X00} + p_{00X0} + p_{000X} \geq 1 - \frac{12\delta^2}{\Gamma^2}$, any probability with more than two X entries is smaller than $\frac{12\delta^2}{\Gamma^2}$. In summary,

$$\frac{\delta}{\Gamma} - 7 \frac{\delta^2}{\Gamma^2} \leq p_{X000}, p_{0X00}, p_{00X0}, p_{000X} \leq \frac{\delta}{\Gamma},$$

after a time of the order $\frac{1}{\delta}$, where the upper bounds are found using Eq. (8). Hence, a steady state is reached where states labelled with one (more than one) X are occupied with probability $\mathcal{O}(\alpha)$ ($\mathcal{O}(\alpha^2)$).

Differential equation for ρ_0

The evolution of $\rho_0 \equiv \rho_{0000}$ is governed by Eq. (??). After a period of the order $\frac{1}{\delta}$, $\dot{\rho}_0 = \gamma' \mathcal{O}(1 + \alpha)$ since $\|\rho_{000X}\| = \frac{\delta}{\Gamma} + \mathcal{O}(\alpha^2)$. Hence, for $\alpha \ll 1$, ρ_0 is approximately constant on time scales that are short compared to γ' . In order to obtain an equation which depends only on ρ_0 , we solve successively the differential equations for $\rho_{i000}, \rho_{0i00}, \rho_{00(ij)0}$ and $\rho_{000(ij)}$. According to Eq. (4),

$$\dot{\rho}_{i000} = -\Gamma \rho_{i000} + \delta A_i \rho_0 A_i^\dagger + \mathfrak{N},$$

where $\mathfrak{N} = \gamma \mathcal{L}(\rho_{i000}) - \delta \rho_{i000} + \delta \sum_k A_i \rho_{k000} A_i^\dagger + \Gamma \sum_{xy=1} T_{xy}(\rho_{i00(xy)})$, which is bounded by $\gamma' \mathcal{O}(\alpha)$. For the first three terms we use $p_{X000} = \mathcal{O}(\alpha)$, the last term can be bounded by $p_{X00X} = \mathcal{O}(\alpha^2)$ such that

$$\rho_{i000}(t) = \rho_{i000}(0) e^{-\Gamma t} + \int_0^t d\tau e^{\Gamma(\tau-t)} \left(\delta A_i \rho_0 A_i^\dagger + \mathfrak{N} \right).$$

The integral $\int_0^t d\tau e^{\Gamma(\tau-t)} \mathfrak{N}$ can be bounded by $\mathcal{O}(\alpha^2)$. The initial term is suppressed by $e^{-\Gamma t}$ and therefore smaller than $\mathcal{O}(\alpha^2)$ after the initial waiting time. Hence,

$$\rho_{i000}(t) = \mathcal{O}(\alpha^2) + \delta \int_0^t d\tau e^{\Gamma(\tau-t)} A_i \rho_0 A_i^\dagger. \quad (9)$$

Since the integral is mainly determined by terms close to $\tau = t$ and ρ_0 varies little on small time intervals, ρ_0 can be assumed to be constant. To prove this, we consider

$$\int_0^t X(\tau, t) d\tau = \int_0^{t'} X(\tau, t) d\tau + \int_{t'}^t X(\tau, t) d\tau, \quad (10)$$

with $X(\tau, t) = e^{\Gamma(\tau-t)} A_i \rho_0 A_i^\dagger$ and $t' = t - \frac{1}{\sqrt{\Gamma \gamma'}} = t - \frac{1}{\gamma'} \sqrt{\alpha}$. Since $\frac{1}{\gamma'}$ is the typical time during which ρ_0 changes, it

is nearly constant during the interval (t, t') . From $\dot{\rho}_0 = \mathcal{O}(\gamma'(1 + \alpha))$, we obtain that for any $t'' \in [t', t]$

$$\rho_0(t'') = \rho_0(t) + \int_t^{t''} d\tau \dot{\rho}_0(\tau) = \rho_0(t) + \mathcal{O}(\sqrt{\alpha}). \quad (11)$$

The integral from 0 to t' in Eq. (10) are suppressed at least by a factor $e^{-\sqrt{\frac{1}{\alpha}}} < \alpha$ ($A_i \rho_0 A_i^\dagger$ is on the order of one). Inserting Eq. (11) in Eq. (9) and using $\delta \int_t^{t'} d\tau e^{\Gamma(\tau-t)} = \frac{\delta}{\Gamma}(1 - e^{-\sqrt{\alpha^{-1}}})$ with $e^{-\sqrt{\alpha^{-1}}} < \alpha$ yields

$$\rho_{i000}(t) = \frac{\delta}{\Gamma} A_i \rho_0(t) A_i^\dagger + \frac{\delta}{\Gamma} \mathcal{O}(\sqrt{\alpha}), \quad (12)$$

which shows that for small α , Alice applies her first POVM with high accuracy.

Next, we consider the evolution of ρ_{0i00} (see Eq. (5)),

$$\dot{\rho}_{0i00} = -\Gamma \rho_{0i00} + \Gamma \rho_{i000} + \mathfrak{N}, \quad (13)$$

where \mathfrak{N} can be bounded by $\gamma' \mathcal{O}(\alpha)$ using $p_{0X00} = \mathcal{O}(\alpha)$, $p_{XX00} = \mathcal{O}(\alpha^2)$ and $p_{0X0X} = \mathcal{O}(\alpha^2)$. Inserting Eq. (12) yields $\rho_{0i00}(t) = \rho_{0i00}(0)e^{-\Gamma t} + \int_0^t d\tau e^{\Gamma(\tau-t)} (\delta A_i \rho_0(\tau) A_i^\dagger + \delta \mathcal{O}(\sqrt{\alpha}) + \mathfrak{N})$. As before, the integral over \mathfrak{N} and the first term can be bounded by $\mathcal{O}(\alpha^2)$ after a waiting time. If the remaining integral is split as in Eq. (10), we obtain one part, where ρ_0 is nearly constant and one vanishing part. The main error is again due to expression (11), leading to

$$\rho_{0i00}(t) = \frac{\delta}{\Gamma} A_i \rho_0(t) A_i^\dagger + \frac{\delta}{\Gamma} \mathcal{O}(\sqrt{\alpha}). \quad (14)$$

Hence, for small α , sending classical information to Bob causes only marginal errors on the main system.

Next, we consider the evolution of $\rho_{00(ij)00}$ (Eq. (6)),

$$\dot{\rho}_{00(ij)0} = -\Gamma \rho_{00(ij)0} + \Gamma B_j^i \rho_{0i00} B_j^{i\dagger} + \mathfrak{N},$$

where \mathfrak{N} can be bounded by $\gamma' \mathcal{O}(\alpha)$ such that

$$\rho_{00(ij)0}(t) = \frac{\delta}{\Gamma} B_j^i A_i \rho_0(t) A_i^\dagger B_j^{i\dagger} + \frac{\delta}{\Gamma} \mathcal{O}(\sqrt{\alpha}),$$

which corresponds to a process, where Bob applies his part of the POVM and writes i, j onto his classical input register. Similarly, Eq. (7) leads to

$$\rho_{000(ij)}(t) = \frac{\delta}{\Gamma} B_j^i A_i \rho_0(t) A_i^\dagger B_j^{i\dagger} + \frac{\delta}{\Gamma} \mathcal{O}(\sqrt{\alpha}),$$

which corresponds to a transfer of the classical measurement results i, j back onto Alice's side. Finally, these results can be applied for calculating ρ_0 ,

$$\dot{\rho}_0 = \gamma \mathcal{L}(\rho_0) - \delta \rho_0 + \Gamma \sum_{ij} T_{ij}(\rho_{000(ij)}) = \gamma \mathcal{L}(\rho_0) - \delta (T'(\rho_0) - \rho_0).$$

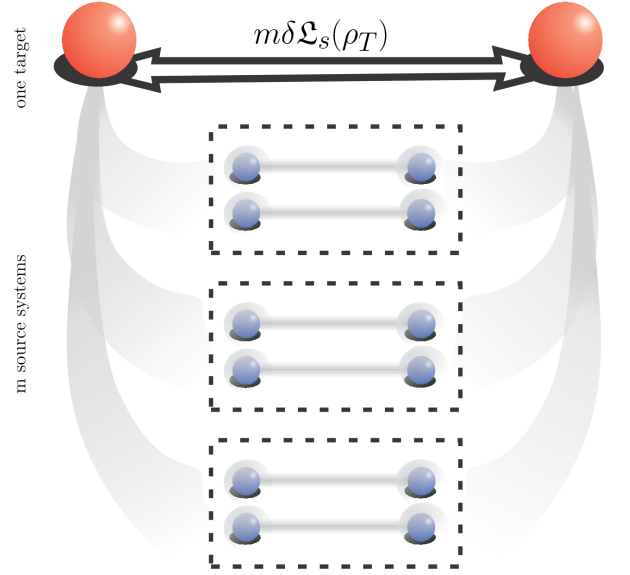


FIG. S.5: (Color online) Stabilization of dissipative protocols against noise acting on the target system by coupling several source systems to the same target.

$T'(\rho) \equiv T(\rho_0) + \mathcal{O}(\sqrt{\alpha})$ represents a noisy version of the desired LOCC map $T(\rho)$. The undesired contribution can be suppressed by choosing α small, i.e. by choosing Γ large enough. The generalization to more than one round of communication is straight forward. By summing over the indices of the corresponding Kraus operators, one obtains equations for the probabilities, which are independent of the POVMs applied in the protocol. From these equations it can be concluded that only the relevant state responsible for the application of the LOCC is populated, while all others are suppressed by a factor of order α^2 after a initial waiting time. By successive integration as shown above, the desired approximation for $\dot{\rho}$ is obtained. Many distillation protocols only require only a small number of rounds to reach high fidelities and often even only one-way communication (half a round) [9,15].

3. STABILIZATION OF DISSIPATIVE DISTILLATION SCHEMES AGAINST ERRORS ACTING ON THE TARGET SYSTEM

In this section, we explain how the distillation schemes presented in Sec. 1.2 and Sec. 4 can be made robust against noise acting on the target system. The same method for stabilization against errors is applicable for both protocols and a wide range of other dissipative schemes, which include classical communication. The basic idea is illustrated in Fig. S.5. A dissipative protocol is run using m blocks of source systems in parallel,

which are all coupled individually to the same target system. This way, a boost effect on the desired dynamics of the target system can be achieved, while the back-action on the source pairs remains unchanged. If sufficiently many source systems are provided, the dynamics on the target system is dominated completely by the desired dynamics. In the following, we explain the application of this method first for schemes of the type described in Sec. 4 and 5 and discuss then briefly the stabilization of scheme I. We start out by considering a target system \mathcal{T} and a source block consisting of n pairs. An entangling dissipative process described by the Lindblad operator $\delta\mathcal{L}(\rho)$ acts on each source pair separately such that each of them is individually driven into an entangled steady state ρ_s . The effective master equation for the target system, which is obtained by tracing out the source system, is given by

$$\dot{\rho}_{\mathcal{T}} = \delta\mathcal{L}_S(\rho_{\mathcal{T}}),$$

where the Lindblad operator $\mathcal{L}_S(\rho_{\mathcal{T}})$ may be time dependent. It does not depend on the state of \mathcal{T} but only on the state of the source system as indicated by the subscript S . Accordingly, the convergence speed at which $\mathcal{L}_S(\rho_{\mathcal{T}})$ converges to a constant operator is given by the rate at which the source system reaches a steady state. The convergence rate of the source system is limited by the rate δ at which the flip operation mapping the quantum states of the source system to \mathcal{T} is performed (see Sec. 1 and Sec. 4).

We assume that m identical source system S_1, \dots, S_m are individually coupled to a single source system \mathcal{T} through the Lindblad operator $\delta \sum_{i=1}^m \mathcal{L}(\rho_{\mathcal{T}, S_i})$, where $\mathcal{L}(\rho_{\mathcal{T}, S_i})$ is a Lindblad operator acting on \mathcal{T} and the i th source system (see Fig. S.5 for a schematic overview). We assume that these operators are identical $\mathcal{L}_{S_i} = \mathcal{L}_S$ such that the dynamics of the target system is governed by the reduced master equation

$$\dot{\rho}_{\mathcal{T}} = \delta \sum_i \mathcal{L}_{S_i}(\rho_{\mathcal{T}}) = m\delta\mathcal{L}_S(\rho_{\mathcal{T}}).$$

This is not generally the case, since the m source systems are coupled to each other through the target system. Due to this indirect coupling, the source systems may evolve differently in time and can reach different steady states, which can be disadvantageous for the evolution of the target system. This is for example the case for the scheme described in Sec. 1.1 which does not include classical communication.

It can be shown that $\mathcal{L}_{S_i}(\rho_{\mathcal{T}}) = \mathcal{L}_S(\rho_{\mathcal{T}})$, if there is no state dependent back-action of \mathcal{T} on the source systems. In this case, the evolution of the reduced density matrix of each source block is independent from the time evolution of the other blocks. This property can be guaranteed by re-initializing the source systems after

each swap operation in a standard state, for example the identity (strict equality requires in principle also that all source systems start from the same initial state). However, different initial states have only an effect on the time evolution in the beginning. The following discussions are only concerned with the steady state of the system, which is independent of the initial conditions). Scheme I including classical communication (see Sec. 1.2) exhibits a weak state dependent back-action. As explained in the end of Sec. 1.2, this can be avoided by applying a twirl [14] on the target system prior to each flip operation. Hence, the stabilization method outlined above is directly applicable to this modified version of the scheme [S4].

By boosting the desired dynamics on the target system, arbitrary high error rates ϵ can be tolerated. For $m\delta \gg \epsilon$, the dynamics governed by the master equation

$$\dot{\rho}_{\mathcal{T}} = m\delta\mathcal{L}_S(\rho_{\mathcal{T}}) + \epsilon\mathcal{L}_{\text{noise}}(\rho_{\mathcal{T}})$$

is dominated by the first term and the steady state is arbitrarily close to the original steady state. In the specific case, where the process $\mathcal{L}_S(\rho_{\mathcal{T}}) = \text{tr}(\rho_{\mathcal{T}})\rho_{\mathcal{T},s} - \rho_{\mathcal{T},s}$ driving the target system into the steady state $\rho_{\mathcal{T},s}$ is counteracted by depolarizing noise ($\text{tr}(\rho_{\mathcal{T}})\mathbb{I} - \rho_{\mathcal{T}}$), the time evolution described by

$$\dot{\rho}_{\mathcal{T}} = m\delta(\text{tr}(\rho_{\mathcal{T}})\rho_{\mathcal{T},s} - \rho_{\mathcal{T}}) + \epsilon(\text{tr}(\rho_{\mathcal{T}})\mathbb{I} - \rho_{\mathcal{T}})$$

leads to the steady state $\rho'_{\mathcal{T},s} = \frac{m\delta\rho_{\mathcal{T},s} + \epsilon\mathbb{I}}{m\delta + \epsilon}$, which can be easily verified by solving the equation $\dot{\rho}_{\mathcal{T}} = 0$. This state is reached exponentially fast with a rate $m\delta + \epsilon$. The same result holds for local depolarizing noise acting on Alice's and Bob's system (see Sec. 4.1) if the steady state $\rho_{\mathcal{T},s}$ is a Werner state. A master equation of this type is solved exactly in the next section.

4. SCHEME II: DISSIPATIVE ENTANGLEMENT DISTILLATION FOR WERNER STATES

In this section, we introduce a second dissipative distillation scheme, which does not rely on entangling processes producing steady states, which are close to pure states, as scheme I presented in Sec. 1. We analyze here a very general model for Werner states [14], which can be solved exactly. Werner states are of the simple form $\rho_W(f) = f\Omega + (1-f)(\mathbb{I} - \Omega)/3$, and are characterized in terms of their fidelity f , which is given by the overlap with the maximally entangled state Ω . Any quantum state can be transformed into a Werner state by twirling [14] without a loss of fidelity. Since a Werner-twirl is a LOCC map, a dissipative protocol can be constructed, which corresponds to the continuous application of a twirl operation on a given system and mapping of the resulting

state to a new pair acting as target system \mathcal{T} by means of a continuous flip procedure (compare Sec. 4.2).

This way, any dissipative process can be modified such that it can be described in terms of a Werner Lindblad operator $E_f(\rho_{\mathcal{T}})$ as used in Secs. 4 and 5, where f is the steady state fidelity of the underlying process. In this sense, the Werner model used here is very general and can be applied in many situations.

4.1 Dissipative entangling model process for a single source pair

A dissipative model process, which produces an arbitrary Werner state as steady state can be modelled by considering two processes, which generate the steady states Ω and \mathbb{I} respectively, where $\mathbb{I} = \mathbb{I}/4$ denotes the normalized identity. Let $|\psi_i\rangle$ denote the four Bell-states, where $|\psi_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, and σ_i the Pauli matrices, where σ_0 is the identity. A master equation which leads to the steady state $\Omega = |\Psi_0\rangle\langle\Psi_0|$ can be constructed using the four jump operators $Q_i = |\psi_0\rangle\langle\psi_i|$, which give rise to the Lindblad term

$$Q(\rho) = \sum_i \mathfrak{L}^{Q_i}(\rho) = \text{tr}(\rho)\Omega - \rho.$$

Similarly, a master equation which leads to the steady state \mathbb{I} is obtained using the jump operators $W_{ij} = \sigma_i \otimes \sigma_j$, which give rise to the Lindblad term

$$W(\rho) = \sum_{ij} \mathfrak{L}^{W_{ij}}(\rho) = \text{tr}(\rho)\mathbb{I} - \rho.$$

Hence, the Werner state $\rho_W(f)$ with fidelity f is the steady state of the time evolution governed by the master equation

$$E_f(\rho) = fQ(\rho) + \frac{1-f}{3}(3W(\rho) - Q(\rho)) = \text{tr}(\rho)\rho_W - \rho.$$

The Lindblad term $E_f(\rho)$ will be used in the following to model the basic entangling process acting on the source systems.

Local depolarizing noise acting on Alice's (Bob's) side is included using the jump operators $S_i = \mathbb{I}_A \otimes \sigma_i$ ($S_i = \sigma_i \otimes \mathbb{I}_B$), such that the corresponding Lindblad terms are given by

$$\begin{aligned} N_{\text{Alice}}(\rho) &= \sum_i \mathfrak{L}^{S_i}(\rho) = \rho_A \otimes \mathbb{I}_B - \rho, \\ N_{\text{Bob}}(\rho) &= \sum_i \mathfrak{L}^{S_i}(\rho) = \mathbb{I}_A \otimes \rho_B - \rho, \end{aligned}$$

where ρ_A (ρ_B) is the reduced density matrix corresponding to Alice's (Bob's) system and \mathbb{I}_A (\mathbb{I}_B) the normalized identity $\mathbb{I}/2$ on Alice's (Bob's) system. This process describes the continuous replacement of the state on Alice's

(Bob's) side by the completely mixed state. The total master equation

$$\dot{\rho} = \gamma E_f(\rho) + \frac{\varepsilon}{2} N(\rho) \quad (15)$$

where $N(\rho) = N_{\text{Alice}}(\rho) + N_{\text{Bob}}(\rho)$, describes the basic entangling process including local noise. This type of equation will be used frequently in the following sections, as it also describes also the evolution of the target systems once the corresponding source systems have reached the steady state.

The steady state of the time evolution described by Eq. (15) is a Werner state

$$\rho_s = \frac{\gamma \rho_W(f) + \varepsilon \mathbb{I}}{\gamma + \varepsilon} \quad (16)$$

with reduced fidelity $f_s = \frac{\gamma f + \varepsilon \frac{1}{4}}{\gamma + \varepsilon}$. The general time dependent solution of the master equation (15) is of the form

$$\rho(t) = \rho_0 g_0(t) + \rho_1 g_1(t) + \rho_2 g_2(t) + \rho_3 g_3(t), \quad (17)$$

where ρ_0 is any initial state, $\rho_1 = \frac{1}{2}(\rho_{0,A} \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \rho_{0,B})$, $\rho_2 = \rho_W(f)$ and $\rho_3 = \mathbb{I} - \rho_0 - \rho_1 - \rho_2$. $\rho_{0,A}$ and $\rho_{0,B}$ are the reduced density matrices of the initial state ρ_0 at Alice's and Bob's side. The functions g_i are given by

$$\begin{aligned} g_0(t) &= e^{-\gamma' t}, \\ g_1(t) &= 2 \left(e^{-\gamma'' t} - e^{-\gamma' t} \right), \\ g_2(t) &= \frac{\gamma}{\gamma'} (1 - e^{-\gamma' t}), \\ g_3(t) &= \frac{2\gamma}{\gamma'} \left(e^{-\gamma' t} - e^{-\gamma'' t} \right) + \frac{\varepsilon}{\gamma'} (e^{\gamma' t} - 2e^{\gamma'' t} + 1), \end{aligned} \quad (18)$$

where $\gamma' = \gamma + \varepsilon$ and $\gamma'' = \gamma + \varepsilon/2$. Note, that the terms which depend on the initial state of the system, i.e. ρ_0 and ρ_1 , are suppressed exponentially fast. The system reaches the steady state given by Eq. (16) exponentially fast with a rate of at least $\gamma + \frac{\varepsilon}{2}$.

In order to verify that Eqs. (17) and (18) are a solution of Eq. (15), Eq. (17) can be used as ansatz. The master equation gives rise to a set of differential equations for the functions g_i with initial conditions $g_0 = 1$ and $g_i = 0$,

$$\begin{aligned} \dot{g}_0 &= -(\gamma + \varepsilon)g_0, \\ \dot{g}_1 &= -(\gamma + \frac{\varepsilon}{2})g_1 + \varepsilon g_0, \\ \dot{g}_2 &= -\varepsilon f_2 + \gamma(g_0 + g_1 + g_2 + g_3), \\ \dot{g}_3 &= -\gamma g_3 + \frac{\varepsilon}{2}g_1 + \varepsilon g_2. \end{aligned} \quad (19)$$

Below, the initial condition $\rho_0 = \mathbb{I}$ will be considered frequently. In this case the solution simplifies to

$$\rho(t) = \rho_s + (\mathbb{I} - \rho_s)e^{-(\gamma + \varepsilon)t}. \quad (20)$$

4.2 Steady state entanglement distillation acting on n source systems

We consider n systems which are subject to the basic entangling process $\gamma E_f(\rho) + \frac{\varepsilon}{2}N(\rho)$ and are driven into the steady state ρ_s as described in Sec. 4.1. These qubit pairs act as source systems for a LOCC distillation operation T_D , which distills one potentially higher entangled state from these copies. The resulting quantum state is mapped to a target pair \mathcal{T} and each source system is re-initialized in the state \mathbb{I} . We do not specify T_D at this point - the solution derived in this section holds for any n to 1 distillation protocol. We start out by considering only deterministic protocols and generalize the results at the end of this section such that probabilistic schemes are also covered. Note that the complete re-initialization of the source systems represents the worst-case situation regarding the back-action of the target system onto the source pairs. This choice allows us solve the model exactly and to provide a lower bound for dissipative distillation schemes of this type.

The continuous distillation procedure explained above is described by the master equation

$$\dot{\rho} = \sum_{i=1}^n \left(\gamma E_f(\rho) + \frac{\varepsilon}{2} N(\rho) \right)_i + \delta_D (T_D(\rho) - \rho), \quad (21)$$

where $(X(\rho))_i$ stands for the dissipative process $X(\rho)$ acting on the i th source system.

In the following, we determine the time evolution and the steady state of the target system. The reduced master equation for \mathcal{T} depends on the steady state of the reduced source system. Therefore, we start by solving the dynamics on the source system. Since the back-action of \mathcal{T} on the source system does not depend on the quantum state of \mathcal{T} , the time evolution of the source pairs can be considered independently from the target system.

For clarity, the reduced states of source and target system are denoted by σ and $\rho_{\mathcal{T}}$ respectively in this section. The reduced master equation for the n source systems is given by

$$\dot{\sigma} = \sum_{i=1}^n \left(\gamma E_f(\sigma) + \frac{\varepsilon}{2} N(\sigma) \right)_i + \delta_D (\text{tr}(\sigma) \mathbb{I}^{\otimes n} - \sigma). \quad (22)$$

The solution of the homogeneous master equation which describes the entangling dynamics for n independent source systems

$$\dot{\sigma}_*(\sigma_0, t) = \sum_{i=1}^n (\gamma E_f(\sigma_*(\sigma_0, t)) + \varepsilon N(\sigma_*(\sigma_0, t)))_i,$$

is already known (see Sec. 4.1) if the initial state is a product state. $\sigma_*(\sigma_0, t)$ denotes the solution of the homogeneous master equation with initial condition $\sigma_*(\sigma_0, t =$

$0) = \sigma_0$. The solution of the inhomogeneous master equation Eq. (22) is given by

$$\begin{aligned} \sigma(t) &= \sigma_*(\sigma_0, t) e^{-\delta_D t} + \delta_D \int_0^t d\tau \sigma_*(\mathbb{I}^{\otimes n}, t - \tau) e^{-\delta_D(t-\tau)}, \\ &= \sigma_*(\sigma_0, t) e^{-\delta_D t} + \delta_D \int_0^t d\tau \sigma_*(\mathbb{I}^{\otimes n}, \tau) e^{-\tau \delta_D}, \end{aligned}$$

with arbitrary initial condition $\sigma(0) = \sigma_0$. This solution can be easily verified by considering the time derivative

$$\begin{aligned} \dot{\sigma}(t) &= -\delta_D \sigma(t) + e^{-\delta_D t} \dot{\sigma}_*(\sigma_0, t) \\ &\quad + e^{-\delta_D t} \partial_t \left[\delta_D \int_0^t d\tau \sigma_*(\mathbb{I}^{\otimes n}, t - \tau) e^{\tau \delta_D} \right]. \end{aligned}$$

Using $\partial_t \int_0^t g(\tau) f(t - \tau) = f(0)g(t) + \int_0^t g(\tau) \dot{f}(t - \tau)$ and $\sigma_*(\mathbb{I}^{\otimes n}, 0) = \mathbb{I}^{\otimes n}$, one obtains

$$\begin{aligned} \dot{\sigma}(t) &= -\delta_D \sigma(t) + e^{-\delta_D t} \dot{\sigma}_*(\sigma_0, t) \\ &\quad + \delta_D \mathbb{I}^{\otimes n} + e^{-\delta_D t} \partial_t \left[\delta_D \int_0^t d\tau \dot{\sigma}_*(\mathbb{I}^{\otimes n}, t - \tau) e^{\tau \delta_D} \right], \end{aligned}$$

which yields Eq. (22). The steady state

$$\sigma_s = \delta_D \int_0^\infty d\tau \sigma_*(\mathbb{I}^{\otimes n}, \tau) e^{-\delta_D \tau} \quad (23)$$

is reached exponentially fast with a rate of at least δ_D . The homogeneous solution $\sigma_*(\mathbb{I}^{\otimes n}, \tau)$ is given by the tensor product of the solution for a single source pair (20),

$$\sigma_*(\mathbb{I}^{\otimes n}, t) = (\rho_s + (\mathbb{I} - \rho_s) e^{-(\gamma + \varepsilon)t})^{\otimes n},$$

such that Eq. (23) can be further simplified

$$\sigma_s = \int_0^1 dx (\rho_s + (\mathbb{I} - \rho_s) x^{\frac{\gamma + \varepsilon}{\delta_D}})^{\otimes n}. \quad (24)$$

Next, we consider the dynamics of the target system \mathcal{T} described by the time dependent master equation

$$\dot{\rho}_{\mathcal{T}} = \delta_D (T_D(\sigma(t)) - \rho_{\mathcal{T}}),$$

which is solved by

$$\rho_{\mathcal{T}}(t) = \rho_{\mathcal{T}}(0) e^{-\delta_D t} + \int_0^t d\tau \delta_D T_D(\sigma(t)) e^{-\delta_D(t-\tau)}$$

with steady state $T_D(\sigma_s)$. The corresponding steady state fidelity can be inferred by integrating over the fidelities that are obtained if a standard distillation protocol is applied such that

$$f_{\text{out}}(f_s) \equiv f_{\text{out}}(f, \varepsilon) = \int_0^1 dx f_D(f_s + (\frac{1}{4} - f_s) x^{\frac{\gamma + \varepsilon}{\delta_D}})^{\otimes n},$$

where Eq. (24) was used.

So far, it has been assumed, that the underlying

distillation protocol T_D is deterministic, such that a distilled state is available whenever it is applied. However, many distillation protocols of interest are probabilistic, i.e., they only succeed some probability $P(\rho)$. If a probabilistic distillation protocol is used, the corresponding map T_D is defined in such a way, that a flip operation is only performed when the distillation was successful, which leads to a state dependent rate in the master equation

$$\dot{\rho}_T = \delta_D P(\sigma(t)) (T_D(\sigma(t)) - \rho_T).$$

Accordingly, the target system is driven into the same steady state as discussed above with a reduced rate. Once the time evolution of the source system has reached a steady state, the dynamics of the target system is determined by the master equation

$$\dot{\rho}_T = \delta_D P(\sigma_s) (\text{tr}(\rho_T) \rho'_s - \rho_T) = \delta_D p E_{f_{\text{out}}}(\rho_T),$$

where ρ'_s is the distilled steady state of the source system. Since ρ'_s is a Werner state, the target system can act as one of n new source systems which drive a new target system into an even more entangled state. This way, the distillation protocol can be iterated in a nested form.

5. CONTINUOUS QUANTUM REPEATERS

The ability to distribute entangled states of high quality over long distances is of vital importance for quantum communication and quantum network related applications in general. As opposed to classical information, quantum information cannot be cloned. Therefore, classical repeater schemes are not applicable in this context and quantum repeater schemes which respect the coherence of quantum states are required [? ? ?]. In quantum repeater protocols, entanglement is first distributed over short distances L_0 with high accuracy. Then neighboring pairs are connected by a teleportation procedure [S5] (entanglement swapping [? ?]) such that entangled links which span a distance $2L_0$ are obtained. In the next step, two neighboring links of length $2L_0$ are connected by entanglement swapping, resulting in entangled pairs which span a distance $4L_0$. This way, an entangled link of length $L = L_0 2^k$ can be established in k iteration steps (compare Fig. 4 in the main text). However, for non-maximally entangled states, entanglement swapping leads to a considerable degradation in the fidelity of the resulting quantum state. Since the distributed entanglement decreases dramatically every time the length of the entangling links is doubled, it can not be distributed over large distances this way. Therefore an entanglement distillation protocol has to be applied after every entanglement swapping procedure before proceeding to the next stage.

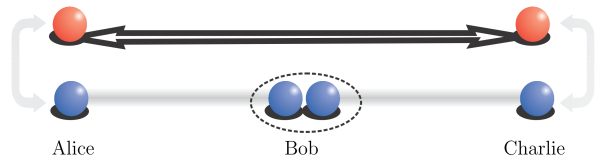


FIG. S.6: (Color online) Continuous entanglement swapping.

In the following, we describe a continuous dissipative quantum repeater scheme, which combines continuous swap and distillation processes in order to generate long-range entangled steady states, while entangling dissipative processes are only required over short distances. To this end, we introduce a continuous swap operation in Sec. 5.1 and explain in Sec. 5.2 how this method can be combined with the distillation scheme presented above (Sec. 4) such that a high-quality entangled link can be established over a large distance as steady state of a continuous dissipative evolution. We conclude this proof-of-principle study by giving a specific example.

5.1 Continuous entanglement swapping

The basic setup for entanglement swapping consists of three nodes aligned on a line, operated by Alice, Bob and Charlie, where Alice and Bob as well as Bob and Charlie share an entangled pair, while the distance between Alice and Charlie is too large for generating an entangled state of high quality (see Fig. 3b in the main text). By performing a teleportation procedure, which requires the measurement of the two qubits at Bob's node and classical communication to Alice and Charlie, as well as local operations on their sides, an entangled link can be established between Alice and Charlie [?].

We consider a setting, where Alice and Bob as well as Bob and Charlie each hold a source pair which is subject to the basic dissipative entangling mechanism considered in Sec. 4, such both pairs are individually driven into the steady state ρ_s . This dynamics is described by the Lindblad term $\gamma_{\text{sw}} \sum_{i=1}^2 (E_f(\rho))_i = \gamma_{\text{sw}} \sum_{i=1}^2 (\text{tr}(\rho) \rho_W - \rho)_i$. As illustrated in Fig. 3b in the main text, the source pairs are coupled to a pair of target qubits shared between Alice and Charlie through the term $\delta_{\text{sw}} (T_{\text{sw}}(\rho) - \rho)$, where the completely positive map T_{sw} corresponds to a flip operation which maps the state resulting from the entanglement swapping procedure to a target system and re-initializes the source systems in the state $\mathbb{I} \otimes \mathbb{I}$ [S7]. Hence, the total master equation is given by

$$\dot{\rho} = \gamma_{\text{sw}} \sum_{i=1}^2 \left(E_f(\rho) + \frac{\varepsilon}{2} N(\rho) \right)_i + \delta_{\text{sw}} (T_{\text{sw}}(\rho) - \rho)$$

and the reduction to the source systems σ yields

$$\dot{\sigma} = \gamma_{\text{sw}} \sum_{i=1}^2 \left(E_f(\rho) + \frac{\varepsilon}{2} N(\rho) \right)_i + \delta_{\text{sw}} (\text{tr}(\sigma) \mathbb{I} \otimes \mathbb{I} - \sigma).$$

The solution of this differential equation (compare Sec. 4.2)

$$\sigma(t) = \sigma_*(\sigma_0, t) e^{-\delta_{\text{sw}} t} + \delta_{\text{sw}} \int_0^t d\tau \sigma_*(\mathbb{I}^{\otimes 2}, t - \tau) e^{-\delta_{\text{sw}}(t - \tau)},$$

where $\sigma_*(\sigma_0, t)$ is the homogeneous solution with initial condition $\sigma(t = 0) = \sigma_0$, shows that the steady state

$$\sigma_s = \int_0^1 dx (\rho_s + (\mathbb{I} - \rho_s) x^{\frac{\gamma_{\text{sw}} + \varepsilon}{\delta_{\text{sw}}}})^{\otimes 2}, \quad (25)$$

where $\rho_s = \frac{\gamma_{\text{sw}} \rho_W(f) + \varepsilon \mathbb{I}}{\gamma_{\text{sw}} + \varepsilon}$, is reached exponentially fast with a rate of at least δ_{sw} .

The time dependent master equation governing the dynamics of the target system is given by

$$\dot{\rho} = \delta_{\text{sw}} (T_{\text{sw}}(\sigma(t)) - \rho_T), \quad (26)$$

where $T_{\text{sw}}(\sigma_s)$ is the steady state of this evolution. According to Eq. (25), the steady state fidelity is given by

$$f_{\text{sw}}(f, \varepsilon) \equiv f_{\text{sw}}(f_s) = \int_0^1 dx F \left(f_s + \left(\frac{1}{4} - f_s \right) x^{\frac{\gamma_{\text{sw}} + \varepsilon}{\delta_{\text{sw}}}} \right),$$

where f_s is the fidelity of the state ρ_s and $F(f) = (1 - 2f + 4f^2)/3$ is the output fidelity of the swap protocol for two input states with fidelity f . A short calculation shows that

$$f_{\text{sw}}(f, \varepsilon) \equiv f_{\text{sw}}(f_s) = 2\gamma_{\text{sw}}^2 \frac{\left(\frac{1}{3}(1 - 2f_s + 4f_s^2) - \frac{1}{4} \right) + \frac{1}{4}}{(2\gamma_{\text{sw}} + \delta_{\text{sw}})(\gamma_{\text{sw}} + \delta_{\text{sw}})}, \quad (27)$$

where f is the fidelity of the state ρ_W and $f_s = \frac{\gamma_{\text{sw}} f + \varepsilon \frac{1}{4}}{\gamma_{\text{sw}} + \varepsilon}$. As discussed in Sec. 3, the scheme can be made robust against noise processes acting on the target system by using m copies of the source systems and coupling them all to the same target state.

5.2 Creation of long-range, high-quality steady state entanglement

The continuous swap operation introduced above (Sec. 5.1), the dissipative distillation protocol explained in Sec. 4 and the method for stabilization against errors acting on target systems (Sec. 3) are the basic building blocks for the dissipative quantum repeater scheme illustrated in Fig. S.7. To begin with, the distance L over which an entangled link has to be established is divided into 2^k segments of length L_0 , as in standard repeater schemes. At each intermediate node, many qubits

are supplied which are subject to local depolarizing noise acting at a rate ε . We assume that each source pair constituting an elementary link of length L_0 is individually driven into a steady state ρ_s of high fidelity by means of an entangling dissipative process of the type discussed in Sec. 4.1., $\gamma E_{f_I}(\rho)$, with high initial steady state fidelity f_I and a rate γ , which is large compared to the noise rate ε . Note that this assumption can also be satisfied starting from dissipative processes leading to a steady state with low fidelity and low γ if distillation and boost processes are applied as discussed above. In the following, we consider an iteration step of the repeater protocol which acts on 2^r entangled source systems, which each span a distance l with fidelity f_l , and produces entangled links of the length $2l$ with fidelity f_{2l} , such that $f_{2l} \geq f_l$. This is illustrated in Fig. S.7, where the entangled source pairs of length l are shown in blue and the yellow target pairs of length $2l$ are depicted in yellow. Each iteration step consist out of the following subroutines, which are illustrated in Fig. S.7b:

- Neighboring source pairs of length l (blue) are connected via a continuous swap operation. The resulting quantum states are written onto target pairs $\mathcal{T}_{\text{sw},i}$ (red). In order to achieve a boost-effect on the targets, this protocol is run on m source systems in parallel.
- A block of n such pairs $\mathcal{T}_{\text{sw},i,j}$, $j = 1, \dots, n$, acts as source system (green) for an distillation process, which maps the resulting quantum state to new target system $\mathcal{T}_{\text{D},i}$ (yellow).
- m of these blocks (green) are needed to achieve a high fidelity of the quantum state of the target systems (yellow).

This iteration step results in entangled links (yellow) which span twice the initial distance and feature a high fidelity as well as a high convergence rate once all source systems have reached the steady state.

In the following, we consider $\delta_D = \delta_{\text{sw}} = \frac{\gamma}{m}$ for simplicity (these parameters can be optimized for a given distillation protocol). The individual levels of the repeater scheme converge seriatim from bottom to top to a steady state. For example, once the source systems of length l (blue) are in a steady state, the reduced master equation for the target system of length $2l$ (yellow) becomes time independent and this system reaches a steady state too. We assume now, that all source pairs of length l (blue) are driven by a time independent master equation of the type discussed in Sec. 4.2, $\dot{\rho} = \gamma E_f(\rho)$ once all underlying systems have reached the steady state. The reduced master equation for the target system $\mathcal{T}_{\text{sw},i}$ (red)

$$\dot{\rho}_{\mathcal{T}_{\text{sw},i}} = \delta m E_{f_{\text{sw}}(f_I, \varepsilon)}(\rho_{\mathcal{T}_{\text{sw},i}}) + \frac{\varepsilon}{2} N(\rho_{\mathcal{T}_{\text{sw},i}}) + \delta (\mathbb{I}^{\otimes n} - \rho_{\mathcal{T}_{\text{sw},i}})$$

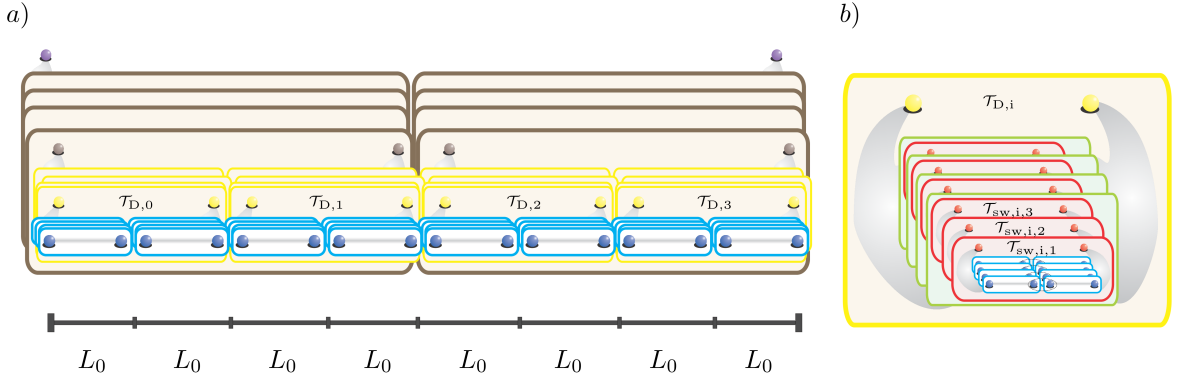


FIG. S.7: (Color online) Dissipative quantum repeater architecture. a) Concatenation of elementary steps, in which the distance over which entanglement is distributed is doubled. b) Illustration of a single iteration step including entanglement swapping and -distillation.

includes local polarizing noise as introduced in Sec. 4.1 and the back-action of the distillation scheme. Note, that the rate of the entangling process, $\delta m = \gamma$ is again high, due to the boost on the target system. The entangling process acting on the target systems of the distillation procedure $\mathcal{T}_{D,i}$ (yellow), $E_{f_{\mathcal{T}_{sw}}}(\rho_{\mathcal{T}_D})$ is determined by the steady state fidelity $f_{\mathcal{T}_{sw}} := f_{sw}(f_I, \varepsilon)$.

A wide range distillation protocols for Werner states [S8] can be used in a continuous form as demonstrated in Sec. 4 (below a specific example is discussed). As explained there, a distillation protocol corresponds to a completely positive map T_D which is described by a Lindblad term $\delta_D (T_D(\rho) - \rho)$. The distillation process is applied continuously for each entangled link and the resulting highly entangled qubit state is flipped to new target pairs $\mathcal{T}_{D,i}$ spanning the same length l .

We consider here a distillation procedure which acts on n entangled source systems and distills one potentially higher entangled pair. Hence, for each of the 2^{r-1} links, n copies $\mathcal{T}_{sw,ij}$, $i \in \{1, 2^{r-1}\}$, $j \in \{1, n\}$ have to be supplied. This situation is sketched in Fig. S.7b, where the target systems $\mathcal{T}_{sw,ij}$ (red), driven by the source pairs (blue), are used as resource for creating a highly entangled steady state of the new target pair (yellow). One source block (shown in green) is sufficient for entanglement distillation, but several of them running in parallel are needed to boost the desired dynamics on the target system. This way, each target pair $\mathcal{T}_{D,i}$ is driven at a rate $m\delta_D = \gamma$ and the total effective master equation for the target systems of the distillation protocol is given by

$$\dot{\rho}_{\mathcal{T}_D} = m\delta_D E_{f_{\mathcal{T}_{sw}}}(\rho) + \frac{\varepsilon}{2}N(\rho).$$

Hence, the resulting steady state fidelity is

$$f_{\mathcal{T}_D} = (f_D(f_{\mathcal{T}_{sw}})) = (f_D(f_{sw}(f_I, \varepsilon))),$$

where $f_D(f_{\mathcal{T}_{sw}})$ is the entanglement distilled from source systems with steady state fidelity $f_{\mathcal{T}_{sw}}$. In order to iterate this process, we require $f_{\mathcal{T}_D} \geq f_I$, which can always

be achieved using a strong entanglement distillation (large n) and high entangling rates γ [S9].

The next iteration step begins with another continuous entangling swapping procedure. Here, the target systems of the distillation scheme act as source systems for the entanglement swapping operation.

Since the total distance L has been divided into 2^k segments ($L = L_0 2^k$), the protocol has to be iterated k times. As explained above, each iteration stage requires $2m^2n$ qubit pairs such that in total $(2m^2n)^k$ source systems are needed. Hence the resources scale with $(L/L_0)^{\log_2(2m^2n)}$ in the distance. We restrict the estimate of the required resources to the number of used qubit pairs, since the other resources scale polynomial in this quantity. As specific example we consider the distribution of an entangled state such that each repeater stage starts with and results in links with fidelity $f = 0.96$. We consider noise acting at a rate $\epsilon = 0.05$, distillation based on $n = 16$ source systems (the distillation protocol is described below) and stabilization of the target pairs by means of $m = 50$ copies of the underlying source blocks and $\gamma \approx 70$. In this example, the required resources scale with $(L/L_0)^{16.4}$.

The entanglement distillation scheme used here is a four-to-one distillation protocol for Werner states [S8] which is applied two times in a nested fashion. Starting from four source states s_1, s_2, s_3, s_4 with fidelity f_{in} , the following operations are performed. First, a bilateral CNOT $_{s_1 \rightarrow s_2}$ operation is applied to the first two pairs (where s_1 is the control and s_2 the target qubit) and s_2 is measured in the computational basis. Then, a Hadamard transformation is performed on both qubits of s_1 . Subsequently, a bilateral CNOT $_{s_1 \rightarrow s_3}$ operation is applied to the first and third pair (where s_1 is the control and s_3 the target qubit) and s_3 is measured in the computational basis. The measurement obtained on Alice's and Bob's side are compared. If their mea-

surement results coincide, the resulting state s_1 is the desired higher entangled state. If not, the "safety-copy" s_4 is used instead. In this event, distillation was not successful and the fidelity has not been increased, but in any case an entangled pair is available. The fidelity of the resulting state is given by

$$f_{out}(f_{in}) = \frac{(1+g)(1+7g^2)}{16P_{succ}},$$

where $g = (4f_{in} - 1)/3$ and $P_{succ} = (1 + g^2 + 2g^3)/4$ is the success probability of this protocol. In the example above, this protocol is applied twice in a concatenated fashion. The second application of the scheme is run using the output states of the first one as input such that $f_D(f_{in}) = f_{out}(f_{out}(f_{in}))$.

We conclude this section with an estimate of the convergence speed of the presented repeater scheme. We start by considering the elementary pairs constituting the entangled links on the lowest level of the scheme. These systems reach the steady state up to a certain high accuracy after a time t_0 . After this time, the dynamics of all systems on the next level is governed to a good approximation by a time independent master equation and converge with high accuracy to the steady state after another time period of length t_0 has elapsed. Convergence of all k levels of the repeater scheme requires therefore a waiting time kt_0 . Since the number of levels used in the scheme scale only logarithmical with the distance, we obtain a very moderate scaling of the convergence time with the distance. Note, that once the whole system is in a steady state, removal of the final long-range entangled pair does not have an effect on the underlying systems which remain in steady state. The repeater protocol put forward here is based on continuous LOCC maps, which represent a particular subset of possible dissipative schemes. We have also presented a distillation protocol (scheme I), without communication which does not fall in this class. However, also the set of dissipative processes assisted by classical communication includes other types of schemes not covered here, which are yet to be explored.

-
- [S1] The variant of scheme I without communication is fundamentally different from scheme II, presented in Sec. 4 and cannot not be formulated in terms master equations of the form $\dot{\rho} \propto (T(\rho) - \rho)$, where $T(\rho)$ is a LOCC channel.
 - [S2] M. Horodecki, P. W. Shor, and M. B. Ruskai, Rev. Math. Phys. **15**, 629 (2003).
 - [S3] The slowest term converges as $\mathcal{O}\left(\frac{t^3 \delta \Gamma^3}{(\Gamma + \delta)} e^{-t(\Gamma + \delta)}\right)$. After an initial waiting time of the order of $\frac{1}{\delta}$ the steady state is reached up to an error of the order $\mathcal{O}\left(\alpha^{-2} e^{-\alpha^{-1}}\right)$.
 - [S4] It can be shown that this back-action has only a small effect on the steady state of the source system, and not accumulate if many source systems are coupled to \mathcal{T} . If the parameter δ is small, the back-action can be bounded several source system can be combined as described above. The strict avoidance of a state depended back-action is enforced only for the sake of clarity.
 - [S5] C. H. Bennett et al, Phys. Rev. Lett. **70**, 1895 (1993).
 - [S6] M. Żukowski, A. Zeillinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. **71**, 4287 (1993); J.-W. Pan, D. Bouwmeester, H. Weinfurter, and A. Zeillinger, Phys. Rev. Lett. **80**, 3891 (1998).
 - [S7] The channel $T_{sw}(\rho)$ corresponds to a three-partite LOCC map [11]. The results in Sec. 2 hold for bi-partite LOCC terms of the form $T(\rho) - \rho$ and can easily extended to multi-partite operations. The continuous realization of the swap scheme is already covered by the derivation for bi-partite LOCC operations presented above, since this protocol can be considered as an effective bi-partite scheme in which Alice and Charlie are treated as one party. Here, a classical one-way channel is used to send two copies of a message at the same time. One copy is sent to Alice and the other one is sent to Charlie.
 - [S8] J. Dehaene, M. Van den Nest, B. De Moor and F. Verstraete, Phys. Rev. A **67**, 022310 (2003).
 - [S9] The error rate ε does not play an important role in this consideration, since the effect of noise on the target systems can be made negligible for large rates γ , without changing the scaling of the protocol. However, this comes at the cost of more resources, that are needed to achieve this rate γ for the initial stage of the repeater protocol, i.e., the number of resources may be multiplied by a large factor, but the scaling with the length stays unchanged.
 - [S10] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046-2052(1996).